Chapter 10. The Second Variation

Weishi Liu
University of Kansas

OUTLINE

Previously, we have focused on necessary conditions for extrema of various functionals some with isoperimetric constraints. The main results include the following equivalent statements:

- The 1st variation of the functional at an extremal in any direction vanishes.
- The Euler-Lagrange system of differential equations is satisfied.

In this chapter, we will discuss

Conditions about whether an extremal is a (local) minimum or maximum,

which is the counterpart of 2nd Derivative Test in Calculus.

§10.1. A quick review of 2nd Derivative Test from Calculus

Let $f : \Omega \to \mathbb{R}$ be smooth where $\Omega \subset \mathbb{R}^n$ with $n = 2$ (simple but relevant).

Given $X = (x_1, x_2)$, $\eta = (\eta_1, \eta_2)$, Taylor's Theorem implies, for ε near zero,

$$\begin{aligned} f(X + \varepsilon\eta) &= f(X) + \varepsilon(f_{x_1}(X)\eta_1 + f_{x_2}(X)\eta_2) \\ &\quad + \frac{\varepsilon^2}{2!}(f_{x_1x_1}(X)\eta_1^2 + 2f_{x_1x_2}(X)\eta_1\eta_2 + f_{x_2x_2}(X)\eta_2^2) + O(\varepsilon^3) \\ &= f(X) + \varepsilon\nabla f(X) \cdot \eta + \frac{\varepsilon^2}{2!}\eta H(X)\eta^T + O(\varepsilon^3), \end{aligned}$$

where $\nabla f = (f_{x_1}, f_{x_2})$ is the gradient vector of f and

$$H = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_1x_2} & f_{x_2x_2} \end{pmatrix}$$

is the (symmetric) Hessian matrix of f.

If $X = (x_1, x_2)$ is a critical point of f, then $\nabla f(X) = 0$, and hence,

$$f(X + \varepsilon\eta) - f(X) = \frac{\varepsilon^2}{2!}\eta H(X)\eta^T + O(\varepsilon^3) = \frac{\varepsilon^2}{2!}\left(\eta H(X)\eta^T + O(\varepsilon)\right).$$

In particular, if $f(X)$ is a local minimum, then $\eta H(X)\eta^T \geq 0$ for $\eta \neq 0$.

In this case, the condition that $\eta H(X)\eta^T \geq 0$ for $\eta \neq 0$ is equivalent to

$$\Delta = f_{x_1x_1}(X)f_{x_2x_2}(X) - f^2_{x_1x_2}(X) \geq 0 \text{ and } f_{x_1x_1}(X) \geq 0.$$

This reminds one the 2D test from Calculus.

It is natural to expect something "similar" for Calculus of Variations.

§10.2. The 2nd variation of a functional at an extremal

1. Derivation of the 2nd variation.

Consider the variational problem

$$J(y) = \int_{x_0}^{x_1} f(x, y, y')dx$$

with fixed boundary conditions $y(x_0) = y_0$ and $y(x_1) = y_1$.

Suppose $y = y(x)$ is an extremal of J in S, i.e., for all η with $\eta(x_0) = \eta(x_1) = 0$,

$$\delta J(\eta, y) = \int_{x_0}^{x_1} \left(\eta f_y + \eta' f_{y'}\right) dx = \int_{x_0}^{x_1} \eta \left(f_y - \frac{d}{dx} f_{y'}\right) dx = 0.$$

<u>Question:</u> Is the extremal $y = y(x)$ a local minimum or maximum?

We need to expand $J(y + \varepsilon\eta)$ with higher order terms in ε.

Step 1. Expansion of the integrand $f(x; y + \varepsilon\eta, y' + \varepsilon\eta')$:

$$f(x, y + \varepsilon\eta, y' + \varepsilon\eta') = f(x, y, y') + \varepsilon\big(f_y\eta + f_{y'}\eta'\big) + \frac{\varepsilon^2}{2!}\big(f_{yy}\eta^2 + 2f_{yy'}\eta\eta' + f_{y'y'}\eta'^2\big) + O(\varepsilon^3).$$

Step 2. Expansion $J(y + \varepsilon\eta)$:

Integrate the expansion in Step 1 over $[x_0, x_1]$ to get

$$J(y + \varepsilon\eta) = J(y) + \varepsilon\delta J(\eta, y) + \frac{\varepsilon^2}{2!}\delta^2 J(\eta, y) + O(\varepsilon^3),$$

where

$$\delta^2 J(\eta, y) = \int_{x_0}^{x_1} \big(f_{yy}\eta^2 + 2f_{yy'}\eta\eta' + f_{y'y'}\eta'^2\big)dx$$

is the second variation of J at $y = y(x)$ along the direction of $\eta = \eta(x)$.

2. A necessary condition for (local) minimum/maximum

Since $y = y(x)$ is an extremal of J, one has $\delta J(\eta, y) = 0$, and hence,

$$J(y + \varepsilon\eta) - J(y) = \frac{\varepsilon^2}{2!}\delta^2 J(\eta, y) + O(\varepsilon^3) = \frac{\varepsilon^2}{2!}\left(\delta^2 J(\eta, y) + O(\varepsilon)\right).$$

Immediately, we have

Theorem. (i) If the extremal $y = y(x)$ of J is a local minimum, then

$$\delta^2 J(\eta, y) \geq 0 \text{ for all } \eta \in H.$$

(ii) Similarly, a necessary condition for $y = y(x)$ to be a local maximum is that

$$\delta^2 J(\eta, y) \leq 0 \text{ for all } \eta \in H.$$

(iii) If $\delta^2 J(\eta, y)$ changes signs, then J cannot have local minima or maxima.

Example. Consider the functional

$$J(y) = \int_0^l (y'^2 - y^2)dx$$

for some fixed $l > 0$ with $y(0) = y(l) = 0$.

A director calculation gives

$$\delta^2 J(\eta, y) = \int_0^l (\eta'^2 - \eta^2)dx.$$

It turns out, with $\eta(0) = \eta(l) = 0$, one always

$$\int_0^l \eta^2 dx \leq \frac{l^2}{\pi^2} \int_0^l \eta'^2 dx$$ – the so-called Poincare inequality,

and hence,

$$\delta^2 J(\eta, y) \geq \left(1 - \frac{l^2}{\pi^2}\right) \int_0^l \eta'^2 dx.$$

Case 1. $l \leq \pi$. In this case, $\delta^2 J(\eta, y) \geq 0$ for any η and any y. Therefore, the necessary condition for a minimum is met for any extremal $y = y(x)$ (so J cannot have a local maximum). In fact, we will show later on that any extremal $y = y(x)$ of J is a local minimum.

Case 2. $l > \pi$. For an integer n, set $\eta_n(x) = \sin \frac{n\pi x}{l}$. Then, $\eta_n(0) = \eta_n(l) = 0$. Using double angle formula, one evaluates that

$$\delta^2 J(\eta_n, y) = \int_0^l (\eta_n'^2 - \eta_n^2) dx = \int_0^l \Big(\frac{n^2\pi^2}{l^2} \cos^2 \frac{n\pi x}{l} - \sin^2 \frac{n\pi x}{l} \Big) dx = \frac{1}{2} \frac{n^2\pi^2 - l^2}{l^2}.$$

Thus, for $n = 1$, $\delta^2 J(\eta_1, y) = \frac{1}{2}\frac{\pi^2 - l^2}{l^2} < 0$; but for any n with $n^2\pi^2 > l^2$, $\delta^2 J(\eta_n, y) = \frac{1}{2}\frac{\pi^2 - l^2}{l^2} > 0$. Therefore, any extremal $y = y(x)$ of J cannot be a local minimum or local maximum.

To this end, recall the 2nd variation of J at y along the direction η is

$$\delta^2 J(\eta, y) = \int_{x_0}^{x_1} \left(f_{yy}\eta^2 + 2f_{yy'}\eta\eta' + f_{y'y'}\eta'^2\right)dx.$$

It is, in general, not practical to decide the sign of $\delta^2 J(\eta, y)$.

In the remaining sections, we will discuss

- simpler but deeper necessary conditions which also motivate sufficient conditions (§10.3 and §10.4); and then,

- a sufficient condition for (local) minima/maxima (§10.5 and §10.6).

§10.3. The Legendre necessary condition

In this section, for a known extremal y of J, we will discuss the so-called Legendre necessary conditions for a local minimum and a local maximum.

Need a closer examination of the condition in terms of the 2nd variation.

Step 1. Rewrite integrand of $\delta^2 J(\eta, y)$ using by-parts: Note that $(\eta^2)' = 2\eta\eta'$ and $\eta(x_0) = \eta(x_1) = 0$. So

$$\int_{x_0}^{x_1} 2 f_{yy'} \eta\eta' dx = \int_{x_0}^{x_1} (\eta^2)' f_{yy'} dx = - \int_{x_0}^{x_1} \eta^2 \frac{d}{dx} f_{yy'} dx,$$

and hence,

$$\delta^2 J(\eta, y) = \int_{x_0}^{x_1} \left[\eta^2 \left(f_{yy} - \frac{d}{dx} f_{yy'} \right) + \eta'^2 f_{y'y'} \right] dx.$$

"Too bad" that one could not rewrite the integral of $\eta'^2 f_{y'y'}$ to convert η'^2 to something with η^2 alike. (You may try though.)

Step 2. Effects of sign of $f_{y'y'}$ on that of $\delta^2 J(\eta, y)$ for ALL η:

For definiteness, we now assume $y = y(x)$ is a local minimum so that

$$\delta^2 J(\eta, y) = \int_{x_0}^{x_1} \left[\eta^2 \left(f_{yy} - \frac{d}{dx} f_{yy'}\right) + \eta'^2 f_{y'y'}\right] dx \geq 0 \text{ for all } \eta \in H.$$

An important observation is that one can

(i) fix a non-zero bound for $|\eta|$, but (ii) make $|\eta'|$ "as large as one wants".

Intuition: a function that oscillates rapidly such as $\sin \frac{x}{\gamma}$ with $\gamma \ll 1$ might do!

This is actually the case (to be shown soon). Thus, one can choose η so that

"the sign of $\eta'^2 f_{y'y'}$ dominates the sign of the integrand of $\delta^2 J(\eta, y)$";

in particular, to have $\delta^2 J(\eta, y) \geq 0$ for ALL η, it is necessary that $f_{y'y'} \geq 0$.

Theorem 10.3.1 (Legendre) If $y = y(x)$ is a local minimum of J in S, then

$$f_{y'y'}(x, y(x), y'(x)) \geq 0 \text{ for all } x \in [x_0, x_1].$$

Proof. Prove by contradiction. Suppose, on the contrary, that $f_{y'y'}(c) = p < 0$ at some $c \in [x_0, x_1]$. For simplicity, we assume $c \in (x_0, x_1)$. By continuity, there is an $\gamma > 0$ so that $[c - \gamma, c + \gamma] \subset [x_0, x_1]$ and $f_{y'y'} < p/2$ for all $x \in (c - \gamma, c + \gamma)$.

Let $k > 2$ be an integer and choose η as

$$\eta(x) = \begin{cases} \sin^{2k} \frac{\pi(x-c)}{\gamma}, & \text{if } x \in [c - \gamma, c + \gamma] \\ 0, & \text{if } x \notin [c - \gamma, c + \gamma]. \end{cases}$$

One then has

$$\eta'(x) = \begin{cases} \frac{2k\pi}{\gamma} \sin^{2k-1} \frac{\pi(x-c)}{\gamma} \cos \frac{\pi(x-c)}{\gamma}, & \text{if } x \in [c - \gamma, c + \gamma] \\ 0, & \text{if } x \notin [c - \gamma, c + \gamma]. \end{cases}$$

[Note: the specific choice of the argument $\frac{\pi(x-c)}{\gamma}$ for the sine function is to make sure that $\eta(c - \gamma) = \eta(c + \gamma) = 0$ and $2k \geq 4$ insures $\eta'(c \pm \gamma) = \eta''(c \pm \gamma) = 0$.]

Now, since $f_{y'y'} < p/2 < 0$ for $x \in (c-\gamma, c+\gamma)$, one has

$$\begin{aligned}\int_{x_0}^{x_1} \eta'^2 f_{y'y'} dx &= \int_{c-\gamma}^{c+\gamma} \eta'^2 f_{y'y'} dx \\ &\leq \frac{p}{2} \frac{4k^2\pi^2}{\gamma^2} \int_{c-\gamma}^{c+\gamma} \sin^{4k-2} \frac{\pi(x-c)}{\gamma} \cos^2 \frac{\pi(x-c)}{\gamma} dx \\ &= \frac{p}{2} \frac{4k^2\pi^2}{\gamma^2} \frac{\gamma}{\pi} \int_{-\pi}^{\pi} \sin^{4k-2} u \cos^2 u du = 2k^2 L_0 \pi \frac{p}{\gamma},\end{aligned}$$

where $L_0 = \int_{-\pi}^{\pi} \sin^{4k-2} u \cos^2 u du > 0$ is a fixed constant.

Note that the other term in the integrand of $\delta^2 J(\eta, y)$ is bounded independent of $\gamma > 0$. Thus, if we take $\gamma > 0$ small enough,

$$\int_{x_0}^{x_1} \eta'^2 f_{y'y'} dx \leq 2k^2 L_0 \pi \frac{p}{\gamma} < 0$$

can be as negative as one wants; in particular, one can have $\delta^2 J(\eta, y) < 0$. This is a contradiction. We thus complete the proof.

Example 1. Consider the functional

$$J(y) = \int_{-1}^{1} x\sqrt{1 + y'^2}dx$$

with $y(-1) = y(1) = 1$.

(It can be checked that $y(x) = 1$ is the only extremal.)

Now, one calculates that

$$f_{y'y'}(x, y(x), y'(x)) = \frac{x}{(1 + y'^2)^{3/2}},$$

which changes signs for $x \in [-1, 1]$.

By Legendre's Theorem, this problem has no local minima nor local maxima.

Example 2. Consider the functional from the previous section

$$J(y) = \int_0^l (y'^2 - y^2)dx$$

for $l > 0$ with $y(0) = y(l) = 0$.

It follows from $f = y'^2 - y^2$ that $f_{y'y'} = 2 > 0$ so the Legendre necessary condition for a local minimum is met.

(i) J cannot have a local maximum, which would require $f_{y'y'} \leq 0$;

(ii) Although the Legendre necessary condition for local minima is met but we already knew from the example in last section that J has a local minimum if $l \leq \pi$ but has NO minima if $l > \pi$.

Remark. If Legendre necessary condition for a local minimum is not met, then the extremal is not a local minimum.

If the necessary condition is met, then one cannot claim the extremal is a local minimum yet – more work is needed.

§10.4. The Jacobi necessary condition

A quick summary for the previous discussion for local minima/maxima:

The starting point is the expansion

$$J(y+\varepsilon\eta) = J(y) + \varepsilon\delta J(\eta, y) + \frac{\varepsilon^2}{2!}\delta^2 J(\eta, y) + O(\varepsilon^3),$$

where $\delta J(\eta, y)$ and $\delta^2 J(\eta, y)$ are the 1st and 2nd variation of J at $y = y(x)$ along the direction $\eta = \eta(x)$, respectively.

If $y = y(x)$ is an extremal, then $\delta J(\eta, y) = 0$ for all η, and hence,

$$(*) \quad J(y+\varepsilon\eta) - J(y) = \frac{\varepsilon^2}{2!}\left(\delta^2 J(\eta, y) + O(\varepsilon)\right).$$

In particular, if $y = y(x)$ is a local minimum, then $\delta^2 J(\eta, y) \geq 0$ for all η.

The Legendre necessary condition is simpler but relies on that

a bounded function that oscillates rapidly can have “large” derivatives.

Recall that

$$\begin{aligned}\delta^2 J(\eta, y) &= \int_{x_0}^{x_1} \left(f_{yy}\eta^2 + 2f_{yy'}\eta\eta' + f_{y'y'}\eta'^2\right) dx \\ &= \int_{x_0}^{x_1} \left(\eta^2\left(f_{yy} - \frac{d}{dx}f_{yy'}\right) + f_{y'y'}\eta'^2\right) dx.\end{aligned}$$

The Legendre necessary condition says one can choose η so that $f_{y'y'}\eta'^2$ "dominates" the integrand so long as $f_{y'y'} \neq 0$. In particular,

$$\delta^2 J(\eta, y) \geq 0 \text{ for all } \eta \implies f_{y'y'} \geq 0 \text{ for all } x \in [x_0, x_1].$$

Thus, a necessary condition for a local minimum is that $f_{y'y'} \geq 0$ for $x \in [x_0, x_1]$.

It is clear that $f_{y'y'} \geq 0$ for $x \in [x_0, x_1]$ does not imply $\delta^2 J(\eta, y) \geq 0$ for all η.

There was an excellent question from the class during the last lecture:

Can $|\eta|$ be made as large as we want while keeping $|\eta'|$ bounded?

The answer is NO since, for any x, there is $x_c \in [x_0, x]$ such that

$$\eta(x) - \eta(x_0) = \eta'(x_c)(x - x_0), \text{ and hence, } |\eta(x)| \le |\eta'(x_c)|(x_1 - x_0).$$

What would happen if the answer were YES? One would get that

$$\delta^2 J(\eta, y) \ge 0 \text{ for all } \eta \implies f_{yy} - \frac{d}{dx} f_{yy'} \ge 0 \text{ for } x \in [x_0, x_1], \text{ and hence,}$$

$$\delta^2 J(\eta, y) \ge 0 \text{ for all } \eta \iff f_{y'y'} \ge 0 \text{ and } f_{yy} - \frac{d}{dx} f_{yy'} \ge 0 \text{ for } x \in [x_0, x_1].$$

"Too bad" the answer to that question is NO!!! That means we have to work harder to find SOMETHING so that, more or less,

$$\delta^2 J(\eta, y) \ge 0 \text{ for all } \eta \iff f_{y'y'} \ge 0 \text{ for } x \in [x_0, x_1] + \text{SOMETHING.}$$

Luckily Jacobi did this for us brilliantly, and we will turn to it now.

For an extremal $y = y(x)$ of J, for easy of notation, we denote

$$p(x) = f_{y'y'}(x, y(x), y'(x)) \text{ and } q(x) = f_{yy} - \frac{d}{dx} f_{yy'}.$$

Then

$$\delta^2 J(\eta, y) = \int_{x_0}^{x_1} \Big(p(x)\eta'^2 + q(x)\eta^2 \Big) dx.$$

• Jacobi's brilliant idea starts from the observation that, for ANY smooth function $w = w(x)$, one has

$$\int_{x_0}^{x_1} (w\eta^2)' dx = 0.$$

Thus, noticing that $(w\eta^2)' = 2w\eta\eta' + w'\eta^2$,

$$\delta^2 J(\eta, y) = \int_{x_0}^{x_1} \Big(p\eta'^2 + 2w\eta\eta' + (w' + q)\eta^2 \Big) dx.$$

For $y = y(x)$ to be a minimum, we know $p(x) \geq 0$ for $x \in [x_0, x_1]$. Let's assume that the strong Legendre necessary condition; that is, $p(x) > 0$ for $x \in [x_0, x_1]$.

Then, the integrand above can be rewritten as

$$\begin{aligned} p\eta'^2 + 2w\eta\eta' + (w'+q)\eta^2 &= p\Big(\eta'^2 + 2\frac{w}{p}\eta\eta' + \frac{w^2}{p^2}\eta^2\Big) + \Big(w' + q - \frac{w^2}{p}\Big)\eta^2 \\ &= p\Big(\eta' + \frac{w}{p}\eta\Big)^2 + \Big(w' + q - \frac{w^2}{p}\Big)\eta^2. \end{aligned}$$

Another observation: If one can find a function $w = w(x)$ so that

$$\text{(R):} \qquad w' + q(x) - \frac{w^2}{p(x)} = 0 \ \text{ for } \ x \in [x_0, x_1],$$

$$\text{then } \ \delta^2 J(\eta, y) = \int_{x_0}^{x_1} p(x)\left(\eta' + \frac{w}{p}\eta\right)^2 dx \geq 0 \ \text{ for any } \ \eta \ \text{ since } \ p(x) > 0.$$

Furthermore, in this case,

$$\delta^2 J(\eta, y) = 0 \iff \eta' + \frac{w}{p}\eta = 0 \iff \eta(x) = 0.$$

The latter follows from uniqueness of solutions of initial value problem and $\eta(x_0) = 0$.

Definition. The 2nd variation $\delta^2 J(\eta, y)$ is called positive definite if $\delta^2 J(\eta, y) > 0$ for $\eta \neq 0$ and $\eta \in H$.

Immediately, if $p(x) = f_{y'y'} > 0$ and the ODE (R) has a solution $w = w(x)$ for $x \in [x_0, x_1]$, then $\delta^2 J(\eta, y)$ is positive definite.

Intuitively, if $\delta^2 J(\eta, y)$ is positive definite, then $y = y(x)$ is a local minimum. We know this is the case in calculus (for finite-dimensional problem). So the ODE (R) is the key. The equation is called a Riccati Equation. It is a 1st order but nonlinear so one does not know if it has a solution defined for $x \in [x_0, x_1]$.

• A standard technique for study of the Riccati Equation (R) is to covert this nonlinear 1st order ODE to a 2nd order linear system as follows.

Introduce $u = u(x)$ for $x \in [x_0, x_1]$ through

$$w(x) = -\frac{p(x)u'(x)}{u(x)} \quad \text{or} \quad u(x) = u_0 e^{-\int_{x_0}^{x} \frac{w(z)}{p(z)} dz} \quad \text{for some } \; u_0.$$

[The minus sign might be missed in the book.]

Then the nonlinear 1st order Riccati equation

$$(R): \qquad w' + q(x) - \frac{w^2}{p(x)} = 0 \;\text{ for }\; x \in [x_0, x_1]$$

is transformed to the 2nd order linear ODE (called the Jacobi Accessory Equation)

$$(J): \quad (p(x)u')' - q(x)u = 0 \;\text{ for }\; x \in [x_0, x_1].$$

We do know that any solution of the linear ODE is always defined for all $x \in [x_0, x_1]$ as long as $p(x) > 0$ and $q(x)$ are continuous for $x \in [x_0, x_1]$.

But, to transform back to $w(x)$ from $u(x)$, one sees that $u(x)$ cannot be zero for any $x \in [x_0, x_1]$. We do not know if that is the case for solutions of equation (J) in general.

The good news is we can have a closer look at this issue since u satisfies a linear ODE, and we will do now.

Recall the 2nd order linear equation (J) always has a general solution of the form

$$u_c(x) = c_1 u_1(x) + c_2 u_2(x),$$

where $u_1(x)$ and $u_2(x)$ are linearly independent solutions of (J).

To state the result on whether or not there is a nowhere vanishing solution $u(x)$ of (J), an important concept is needed.

We say $x^* \neq x_0$ is a conjugate point to x_0 if the equation (J) has a non-trivial solution $u(x)$ such that $u(x_0) = u(x^*) = 0$.

The key result is

Theorem. Suppose $p(x) > 0$ and there are no conjugate points to x_0 in $(x_0, x_1]$. Then the Jacobi Accessory Equation (J) has a solution $u = u(x)$ such that $u(x) \neq 0$ for all $x \in [x_0, x_1]$.

As a consequence, in this case, the Riccati equation will have a solution $w(x)$ for $x \in [x_0, x_1]$, and hence, $\delta^2 J(\eta, y)$ is positive definite.

Example 1. Let $h(x) > 0$ for $x \in [x_0, x_1]$. Consider

$$J(y) = \int_{x_0}^{x_1} h(x) y'^2 dx$$

with fixed end points.

For any extremal $y = y(x)$, one finds that

$$p(x) = f_{y'y'} = 2h(x) > 0 \text{ and } q(x) = f_{yy} - \frac{d}{dx} f_{yy'} = 0.$$

Thus, the Jacobi Accessory Equation is $(2h(x)u')' = 0$, which has a general solution

$$u_c(x) = c_1 \int_{x_0}^{x} h^{-1}(z) dz + c_2.$$

For conjugate points to x_0, we are looking for nonzero solutions $u(x)$ with $u(x_0) = 0$. They are given by $c_0 = 0$ or $u(x) = c_1 \int_{x_0}^{x} h^{-1}(z) dz$ with $c_1 \neq 0$. But such a function $u(x)$ does not vanish if $x \neq x_0$. Thus there are no conjugate points to x_0 in $(x_0, x_1]$. By the theorem, we claim that $\delta^2 J(\eta, y)$ is positive definite.

Example 2. Consider again the functional

$$J(y) = \int_0^l (y'^2 - y^2) dx$$

for $l > 0$ with $y(0) = y(l) = 0$.

It follows that $p(x) = f_{y'y'} = 2 > 0$ and $q(x) = f_{yy} - \frac{d}{dx} f_{yy'} = -2$. The Jacobi Accessory Equation is then $(pu')' - q(x)u = 0$ or $u'' + u = 0$, which has a general solution

$$u_c(x) = c_1 \sin x + c_2 \cos x.$$

Nonzero solutions that vanish at $x_0 = 0$ are $u(x) = c_1 \sin x$ with $c_1 \neq 0$. Note that $u(x) = 0$ at $x = k\pi$ for any integer k. Thus, if $l < \pi$, then there are no conjugate points to $x_0 = 0$ in $[x_0, x_1] = [0, l]$, and hence, for any extremal of J, the second variation is positive definite; if $l > \pi$, then $\pi \in (0, l]$ is a conjugate point to $x_0 = 0$, which is consistent to the known fact that the second variation is not positive definite.

Again a quick summary of previous discussions for local minima/maxima:

Given an extremal $y = y(x) \in S$, for any $\eta = \eta(x) \in H$ and small ε, one has

$$(*) \quad J(y + \varepsilon\eta) - J(y) = \frac{\varepsilon^2}{2!}\left[\delta^2 J(\eta, y) + O(\varepsilon)\right].$$

Clear: If $y = y(x)$ is a local minimum, then $\delta^2 J(\eta, y) \geq 0$ for all η.

Guess: If $\delta^2 J(\eta, y) > 0$ for all $\eta \neq 0$, then $y = y(x)$ is a local minimum.

$$\text{Recall:} \quad \delta^2 J(\eta, y) = \int_{x_0}^{x_1} \left(\eta^2\left(f_{yy} - \frac{d}{dx} f_{yy'}\right) + f_{y'y'}\eta'^2\right) dx.$$

$$\text{Legendre:} \quad \delta^2 J(\eta, y) \geq 0 \ \text{ for all } \ \eta \implies f_{y'y'} \geq 0 \ \text{ for } \ x \in [x_0, x_1].$$

Identify conditions, in addition to $f_{y'y'} > 0$, that imply $\delta^2 J(\eta, y) > 0$ for $\eta \neq 0$.

Jacobi found, for any $w = w(x)$, one always has

$$\delta^2 J(\eta, y) = \int_{x_0}^{x_1} \left[p\Big(\eta' + \frac{w}{p}\eta\Big)^2 + \Big(w' + q - \frac{w^2}{p}\Big)\eta^2 \right] dx,$$

and hence, if $w = w(x)$ satisfies the nonlinear 1st order Riccati equation

$$(R): \qquad w' + q(x) - \frac{w^2}{p(x)} = 0 \ \text{ for } \ x \in [x_0, x_1],$$

then $\delta^2 J(\eta, y)$ is positive definite.

With $w(x) = -\dfrac{p(x)u'(x)}{u(x)}$, (R) becomes the Jacobi Accessory Equation

$$(J): \quad (p(x)u')' - q(x)u = 0 \ \text{ for } \ x \in [x_0, x_1].$$

If $p(x) > 0$ and there is no conjugate point to x_0 in $(x_0, x_1]$, then (J) has solutions with $u(x) \neq 0$ for $x \in [x_0, x_1]$ so (R) has a solution $w = w(x)$ for $x \in [x_0, x_1]$, and hence, $\delta^2 J(\eta, y)$ is positive definite.

Finally we have the following result that would yield Jacobi Necessary condition as a corollary.

Theorem. Let f be a smooth function and let $y = y(x)$ be a smooth extremal for the functional

$$J(y) = \int_{x_0}^{x_1} f(x, y, y')dx$$

with fixed end points such that $p(x) = f_{y'y'} > 0$ for $x \in [x_0, x_1]$.

I. If $\delta^2 J(\eta, y)$ is positive definite, then there is no conjugate point to x_0 in $(x_0, x_1]$.

II. If $\delta^2 J(\eta, y) \geq 0$ for all $\eta \in H$, then there is no conjugate point to x_0 in (x_0, x_1).

Remark. Note that $\delta^2 J(0, y) = 0$.

Recall: $\delta^2 J(\eta, y)$ is positive definite if $\delta^2 J(\eta, y) > 0$ for all $\eta \neq 0 \in H$.

The statement that "$\delta^2 J(\eta, y) \geq 0$ for all $\eta \in H$" allows the possibility that "$\delta^2 J(\eta, y) = 0$ for some $\eta \neq 0 \in H$".

Proof of Statement I.

Step 1. First of all, we show that $x = x_1$ cannot be conjugate to x_0. Suppose, on the contrary, x_1 is conjugate to x_0. Then there is a function $u_* = u_*(x) \neq 0$ with $u_*(x_0) = u_*(x_1) = 0$ satisfies (J); that is,

$$(p(x)u_*')' - q(x)u_* = 0.$$

Now, multiply above by u_* and integrate to get, with an application of integral by parts,

$$p(x)u_*'(x)u_*(x)|_{x_0}^{x_1} - \int_{x_0}^{x_1} p(x)u_*'^2 dx - \int_{x_0}^{x_1} q(x)u_*^2 dx = 0$$

or, $\delta^2 J(u_*, y) = 0$ for $u_* \neq 0 \in H$. This contradicts the assumption of I.

Step 2. To show there is no conjugate points to x_0 in (x_0, x_1), one introduces, for any $\mu \in [0, 1]$, a new functional in η:

$$K_\mu(\eta) = \mu\delta^2 J(\eta, y) + (1 - \mu)P(\eta),$$

where

$$P(\eta) = \int_{x_0}^{x_1} \eta'^2 dx.$$

It is easy to see that the functional P has no conjugate points in $(x_0, x_1]$ and also it is easy to see that, for any $\mu \in [0, 1]$, $K_\mu(\eta)$ is positive definite.

The Jacobi Accessory Equation associated to K_μ is

$$(J)_\mu : \quad [(\mu p(x) + 1 - \mu)u']' - \mu q(x)u = 0.$$

We know that $\mu p(x) + 1 - \mu > 0$ for $\mu \in [0, 1]$ since $p > 0$ (by assumption). Therefore the solution $u(x; \mu)$ with $u(x_0; \mu) = 0$ and $u'(x_0; \mu) = 1$ depends on μ continuously for $x \in (x_0, x_1]$.

For $\mu = 0$, $(J)_0$ is $u'' = 0$, and hence $u(x; 0) = x - x_0$, which has no conjugate points in (x_0, x_1).

Suppose, on the contrary, for $\mu = 1$, there is a conjugate point $x^* \in (x_0, x_1]$, that is, $u(x^*, 1) = 0$. One can conclude that there is $\mu_0 \in (0, 1)$ so that the corresponding solution $u(x; \mu_0)$ vanishes at $x = x_1$; that is, $(J)_{\mu_0}$ has a nonzero solution $u(x; \mu_0)$ with $u(x_0; \mu_0) = u(x_1; \mu_0) = 0$ (particularly, $u(x; \mu_0) \neq 0 \in H$).

Multiply $u(x;\mu_0)$ on $(J)_{\mu_0}$ and integrate over $[x_0, x_1]$ to get

$$\int_{x_0}^{x_1} \Big((\mu_0 p(x) + 1 - \mu_0) u'^2 + \mu_0 q(x) u^2 \Big) dx = 0$$

or $K_{\mu_0}(\eta) = \mu_0 \delta^2 J(\eta, y) + (1 - \mu_0) P(\eta) = 0$ with $\eta(x) = u(x;\mu_0) \neq 0$ and $\eta \in H$. This contradicts to that $\delta^2 J(\eta, y) > 0$ and $P(\eta) > 0$ for $\eta \neq 0 \in H$.

<u>Proof of Statement II.</u> The same procedure in Step 2 works for this case. But Step 1 will not so. Thus one can only conclude that there is no conjugate points in (x_0, x_1).

Remark. The following statements are equivalent.

(i) There is no conjugate point to x_0 in $(x_0, x_1]$.

(ii) The solution $u = u(x)$ of the initial value problem

$$(p(x)u')' - q(x)u = 0, \quad u(x_0) = 0 \ \text{ and } \ u'(x_0) = 1$$

has no zero in $(x_0, x_1]$.

Recall that, If $y = y(x)$ is a local minimum, then $\delta^2 J(\eta, y) \geq 0$ for all $\eta \in H$. We now have the Jacobi Necessary Condition as a consequence of Statement II in the previous theorem.

Corollary. (Jacobi Necessary Condition) Let f be a smooth function and let $y = y(x))$ be a smooth extremal for the functional

$$J(y) = \int_{x_0}^{x_1} f(x, y, y')dx$$

with fixed end points such that $p(x) = f_{y'y'} > 0$ for $x \in [x_0, x_1]$. If $y = y(x)$ is a local minimum, then there is no conjugate point to x_0 in (x_0, x_1).

It turns out, under the strong Legendre condition $p(x) > 0$ for $x \in [x_0, x_1]$,

if there is no conjugate point to x_0 in $(x_0, x_1]$, then $y = y(x)$ is a local minimum.

Note that the difference in the statements: one is the open interval (x_0, x_1) and the other is the half-closed interval $(x_0, x_1]$.

§10.5. A sufficient condition

A summary of previous discussions for local minima/maxima:

Given an extremal $y = y(x) \in S$, for any $\eta = \eta(x) \in H$ and small ε, one has

$$(*) \quad J(y + \varepsilon\eta) - J(y) = \frac{\varepsilon^2}{2!}\left[\delta^2 J(\eta, y) + O(\varepsilon)\right].$$

Suppose $p(x) = f_{y'y'} > 0$ for $x \in [x_0, x_1]$. Then,

$\delta^2 J(\eta, y) \geq 0$ for all $\eta \in H \Longrightarrow$ No conjugate points to x_0 in (x_0, x_1).

No conjugate points to x_0 in $(x_0, x_1] \Longleftrightarrow \delta^2 J(\eta, y)$ is positive definite.

Guess: If $\delta^2 J(\eta, y)$ is positive definite, then $y = y(x)$ is a local minimum.

This is indeed correct; that is, one has a sufficient condition for local minima, Theorem. Let f be a smooth function and let $y = y(x)$ be a smooth extremal for the functional

$$J(y) = \int_{x_0}^{x_1} f(x, y, y')dx$$

with fixed end points such that $p(x) = f_{y'y'} > 0$ for $x \in [x_0, x_1]$. If there is no conjugate point to x_0 in $(x_0, x_1]$ (or equivalently, $\delta^2 J(\eta, y)$ is positive definite), then $y = y(x)$ is a local minimum.

We will skip the proof but want to emphasize a key point.

Recall that given an extremal $y = y(x) \in S$, for any $\eta = \eta(x) \in H$ and small ε, one has

$$(*) \quad J(y + \varepsilon\eta) - J(y) = \frac{\varepsilon^2}{2!}\left[\delta^2 J(\eta, y) + O(\varepsilon)\right].$$

To justify the claim, one needs to show that the positive definite $\delta^2 J(\eta, y)$ can control the term $O(\varepsilon)$ inside the bracket. This is not a trivial task since, for $\eta = 0$, $\delta^2 J(0, y) = 0$, and hence, $\delta^2 J(\eta, y)$ can be made as small as one wants by choosing η appropriately. The point is that $O(\varepsilon)$ also depends on η, which needs Taylor expansion up to $O(\varepsilon^3)$ (see Exercise #2 in Section 10.2), and CAN be controlled by $\delta^2 J(\eta, y)$.

Example 1. Consider

$$J(y) = \int_1^2 x^3 y'^2 dx, \quad y(1) = 0,\ y(2) = 3.$$

The problem has a unique extremal given by

$$y(x) = 4 - \frac{4}{x^2}.$$

Note that $p(x) = f_{y'y'} = 2x^3 > 0$ for $x \in [1, 2]$ and $q(x) = 0$.

The Jacobi equation is $(2x^3u')' = 0$. The solution with $u(1) = 0$ and $u'(1) = 1$ is $u(x) = \frac{1}{2} - \frac{1}{2x^2}$, which has no zero in $(1, 2]$. Therefore, the extremal $y = y(x)$ is a local minimum.

Example 2. Consider

$$J(y) = \int_0^{\pi} \left(y \sin x - y'^2 + 2yy'\right) dx \quad \text{with fixed end points.}$$

One can check that a general solution of the EL is

$$y(x) = c_1 x + c_2 + \frac{1}{2} \sin x.$$

Note that $p(x) = -2 < 0$ for $x \in [0, \pi]$ (suggesting local maxima) and $q(x) = 0$.

The solution of the IVP

$$(-2u')' = 0 \quad \text{with} \quad u(0) = 0 \quad \text{and} \quad u'(0) = 1$$

is $u(x) = x$, which is nonzero for $x \in (0, \pi]$. Thus, the extremal is a local maximum.

§10.6. More on conjugate points

1. From a general solution of EL to a general solution of Jacobi equation.

Consider

$$J(y) = \int_{x_0}^{x_1} f(x, y, y')dx \quad \text{with fixed end points.}$$

Theorem. Suppose $y = y(x; c_1, c_2)$ is a general solution of the EL; that is, for any c_1 and c_2,

$$\frac{d}{dx} f_{y'}\big(x, y(x; c_1, c_2), y'(x; c_1, c_2)\big) - f_y\big(x, y(x; c_1, c_2), y'(x; c_1, c_2)\big) = 0.$$

Then,

$$u_1(x) = \frac{\partial y}{\partial c_1}(x; c_1, c_2) \quad \text{and} \quad u_2(x) = \frac{\partial y}{\partial c_2}(x; c_1, c_2)$$

are solutions of the Jacobi Accessory Equation

$$(p(x)u')' - q(x)u = 0.$$

Proof. Take the partial derivative with respect to c_1 from the EL equation to get

$$\begin{aligned}
&\frac{\partial}{\partial c_1}\left[\frac{d}{dx} f_{y'}\big(x, y(x;c_1,c_2), y'(x;c_1,c_2)\big) - f_y\big(x, y(x;c_1,c_2), y'(x;c_1,c_2)\big)\right] \\
&= \frac{d}{dx}\left[\frac{\partial}{\partial c_1} f_{y'}\big(x, y(x;c_1,c_2), y'(x;c_1,c_2)\big)\right] - \frac{\partial}{\partial c_1} f_y\big(x, y(x;c_1,c_2), y'(x;c_1,c_2)\big) \\
&= \frac{d}{dx}\left[f_{y'y}\frac{\partial y}{\partial c_1} + f_{y'y'}\frac{\partial y'}{\partial c_1}\right] - \left[f_{yy}\frac{\partial y}{\partial c_1} + f_{yy'}\frac{\partial y'}{\partial c_1}\right] = 0.
\end{aligned}$$

With $u_1 = \partial_{c_1} y$, one has $u_1' = \partial_{c_1} y'$, and hence,

$$\begin{aligned}
0 &= \frac{d}{dx}\left[f_{y'y}u_1 + f_{y'y'}u_1'\right] - \left[f_{yy}u_1 + f_{yy'}u_1'\right] \\
&= \frac{d}{dx} f_{yy'} u_1 + f_{yy'}u_1' + (f_{y'y'}u_1')' - f_{yy}u_1 - f_{yy'}u_1' \\
&= (p(x)u_1')' - q(x)u_1.
\end{aligned}$$

Thus, u_1 is a solution of the Jacobi Accessory Equation. Similarly, so is u_2.

Example 1. Consider

$$J(y) = \int_0^\pi \left(y \sin x - y'^2 + 2yy'\right) dx \text{ with fixed end points.}$$

One can check that a general solution of the EL is

$$y(x) = c_1 x + c_2 + \frac{1}{2} \sin x.$$

Thus, $u_1(x) = y_{c_1} = x$ and $u_2(x) = y_{c_2} = 1$ are two solutions of the Jacobi equation, and hence, a general solution is

$$u(x) = Au_1(x) + Bu_2(x) = Ax + B \text{ with } u'(x) = A.$$

The solution with the initial values $u(0) = 0$ and $u'(0) = 1$ is $u(x) = x$ (as was shown in the previous part). Therefore, there is no conjugate point to $x = 0$ in $[0, \pi]$. So the extremal is a local maximum since $p(x) = -2 < 0$.

Example 2. Consider $J(y) = \int_0^l \left(y'^2 - y^2\right) dx$ with $y(0) = y(l) = 0$.

The EL equation is $y'' + y = 0$, which has a general solution

$$y(x) = c_1 \cos x + c_2 \sin x.$$

Thus, $u_1(x) = \cos x$ and $u_2(x) = \sin x$ are two solution of the Jacobi Accessory Equation, and hence, a general solution is

$$u(x) = Au_1(x) + Bu_2(x) = A\cos x + B\sin x \quad \text{with} \quad u'(x) = -A\sin x + B\cos x.$$

The initial values $u(0) = 0$ and $u'(0) = 1$ yield $A = 0$ and $B = 1$. So the corresponding solution is $u(x) = \sin x$, which has zeros $x = k\pi$. In particular,

if $l < \pi$, then there is no conjugate point to $x = 0$ in $(0, l]$, and hence, the extremal is a local minimum;

if $l > \pi$, then $x = \pi \in (0, l)$ is a conjugate to $x = 0$, and hence, the extremal is not a local minimum.

2. No x in f.

Recall that $H(y,y') = y'f_{y'} - f$ is a 1st integral for the EL of an extremal.

Theorem. If there is no x in f and $u = u(x)$ with $u(x_0) = 0$ and $u'(x_0) = 1$ is the principal solution of the Jacobi equation associated to an extremal $y = y(x)$. Then

$$H_{y'}u' + H_y u = H_{y'}(x_0, y(x_0), y'(x_0)),$$

which is a first order linear ODE.

Proof. It suffices to show that $H_{y'}u' + H_y u$ is a constant. Note that $H_{y'} = y'f_{y'y'}$ and $H_y = y'f_{yy'} - f_y$. So $H_{y'}u' + H_y u = y'f_{y'y'}u' + (y'f_{yy'} - f_y)u$.

Take the derivative with respect to x to get

$$\begin{aligned} & y'(f_{y'y'}u')' + y''f_{y'y'}u' - y'\left(f_{yy} - \frac{d}{dx}f_{yy'}\right)u + (y'f_{yy'} - f_y)u' \\ &= y'\Big((pu')' - q(x)u\Big) + \left(y''f_{y'y'} + y'f_{yy'} - f_y\right)u'. \end{aligned}$$

Note $(pu')' - q(x)u = 0$ is Jacobi equation and $y''f_{y'y'} + y'f_{yy'} - f_y = 0$ is EL.