

Linear stability of the sub-to-super inviscid transonic stationary wave for gas flow in a nozzle of varying area

John M. Hong,^{*} Cheng-Hsiung Hsu,[†]

Ying-Chieh Lin[‡] and Weishi Liu[§]

Abstract

In this work we consider the linear stability of a sub-to-super inviscid transonic stationary wave of a one-dimensional model of isentropic compressible flows through a nozzle of varying area. This sub-to-super inviscid transonic stationary wave is newly founded by the authors using the geometric singular perturbation theory. The main result of this work is to show that the sub-to-super inviscid transonic stationary wave is physically relevant in the sense that it is L^∞ linearly stable on any bounded space interval as long as its velocity is greater than $1/\sqrt{2}$ of the sound speed.

^{*}Department of Mathematics, National Central University, Chung-Li 32001, Taiwan, E-mail: jhong@math.ncu.edu.tw. Partially supported by National Science Council of Taiwan.

[†]Department of Mathematics, National Central University, Chung-Li 32001, Taiwan, E-mail: chhsu@math.ncu.edu.tw. Partially supported by National Science Council and NCTS of Taiwan.

[‡]Department of Mathematics, National Central University, Chung-Li 32001, Taiwan, E-mail: linyj@math.ncu.edu.tw.

[§]Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA, E-mail: wliu@math.ku.edu. Partially supported by National Science Foundation of USA Grant DMS-0807327 and University of Kansas GRF 2301264-003.

1 Introduction

The following system

$$\begin{cases} (a\rho)_t + (a\rho u)_x = 0, \\ (a\rho u)_t + (a\rho u^2)_x + a(P(\rho))_x = 0 \end{cases} \quad (1.1)$$

is a well-known one-dimensional model of isentropic compressible flow through a narrow nozzle with variable cross-section area (see [2, 3, 5, 14–18, 21–24] etc.). Here ρ , u , P and $a = a(x)$ are the density, velocity, pressure of the gas and the area of the cross-section at x of the rotationally symmetric tube of the nozzle. The pressure P is assumed to be a given function of the density ρ .

Including the equation for the energy E , system (1.1) becomes

$$\frac{\partial w}{\partial t} + \frac{\partial f(w)}{\partial x} = g(x, w), \quad (1.2)$$

where $w = (\rho, \rho u, \rho E)$, $f(w) = (\rho u, \rho u^2 + P, \rho u E + P u)$ and

$$g(x, w) = -\frac{a_x(x)}{a(x)}(\rho u, \rho u^2, \rho u E + P u).$$

Note that the first two equations are decoupled from the third equation.

In [16], T.P. Liu constructed global solutions of the initial value problem for general quasilinear strictly hyperbolic systems of form (1.2) and studied their asymptotic behaviors. Roughly speaking, it was shown that, for an initial data $w_0(x)$ and for w uniformly close to w_0 , if all eigenvalues $\lambda_j(w)$ of f_w are nonzero and the L^1 -norm of g and g_w are small, then a global solution exists and tends pointwise to a steady-state solution of (1.2). In particular, the stability of supersonic and subsonic waves is established.

In [17], T.P. Liu then focused on *transonic* waves of gas flow in a nozzle of varying area via the model (1.1) or (1.2) with the specific f and g . He constructed solutions, under various asymptotic conditions on w , of the steady-state system

$$\frac{\partial f(w)}{\partial x} = g(x, w). \quad (1.3)$$

In particular, solutions demonstrated significant qualitative differences between a contracting nozzle (for example, $a_x(x) < 0$ for $0 < x < 1$ and $a_x(x) \equiv 0$ for $x \notin (0, 1)$) and an expanding nozzle (for example, $a_x(x) > 0$ for $0 < x < 1$ and $a_x(x) \equiv 0$ for $x \notin (0, 1)$). Asymptotic states along a nozzle that contracts and then expands ($a_x(x) < 0$ for $-1 < x < 0$, $a_x(x) > 0$ for $0 < x < 1$ and $a_x(x) \equiv 0$ for $x \notin [-1, 1]$) are also examined to exhibit a number of interesting phenomena including the choking phenomenon. In [18], stability of transonic flows was

examined and it was shown that, flows along expanding nozzles are stable and those with standing shock waves along a contracting nozzle are nonlinearly unstable.

Steady nozzle flows were also investigated numerically by Chakravarthy & Osher [1], Embid et al [5], Shubin et al [21], Smith [24], etc. For standing waves of (1.2), the conservation of energy is a consequence of the first two conservation laws known as Bernoulli law (see [5, 10], etc.).

Recently, using the geometric singular perturbation approach, we examined stationary waves of system (1.1) and their viscous profiles via the viscous regularization. More precisely, in [11], we consider the system

$$\begin{cases} (a\rho)_t + (a\rho u)_x = 0, \\ (a\rho u)_t + (a\rho u^2)_x + a(P(\rho))_x = \varepsilon(au_x)_x, \end{cases} \quad (1.4)$$

where $\varepsilon > 0$ is the viscosity coefficient. Suppose the following assumptions hold:

$$(A-1) \ P(0) = 0, \ P'(\rho) > 0 \text{ and } P''(\rho) \geq 0 \text{ for } \rho > 0 \quad (1.5)$$

and

$$(A-2) \ x a_x(x) > 0 \text{ for } x \neq 0, \ a_{xx}(0) > 0, \ a(x) \rightarrow a_{\pm} \text{ as } x \rightarrow \pm\infty; \quad (1.6)$$

that is, the nozzle is first contracting then expanding. Note that, for polytropic gases, the pressure is $P(\rho) = \theta\rho^\gamma$ for some $\theta > 0$ and $1 \leq \gamma \leq 5/3$ and satisfies condition in (A-1). For such a contracting-expanding nozzle, a detailed and rather complete classification of stationary waves that admit viscous profiles was provided. Quite interestingly, in [11], a smooth *sub-to-super inviscid transonic* stationary wave was constructed for the inviscid system (1.1). More precisely, we proved that, for any $m > 0$, there is a unique quadruple $(\rho_-, u_-, \rho_+, u_+)$ with $m = a_{\pm}\rho_{\pm}u_{\pm}$ so that $u_- < \sqrt{P'(\rho_-)}$ and $u_+ > \sqrt{P'(\rho_+)}$, and a unique smooth stationary wave $(\rho(x; 0), u(x; 0))$ of (1.1) such that $(\rho(x), u(x)) \rightarrow (\rho_{\pm}, u_{\pm})$ as $x \rightarrow \pm\infty$ and, only at $x = 0$, $u(x; 0) = \sqrt{P'(\rho(x; 0))}$. Thus, this smooth stationary wave $(\rho(x; 0), u(x; 0))$ is subsonic for $x < 0$, sonic at $x = 0$ and supersonic for $x > 0$. We also showed that the sub-to-super inviscid transonic stationary wave admits viscous profiles $(\rho(x; \varepsilon), u(x; \varepsilon))$ for system (1.4), which is called a *sub-to-super viscous transonic* stationary wave. Smooth super-to-sub transonic waves, super-to-sub transonic with standing shocks as well as steady-states with a portion of a sub-to-super transonic waves were also constructed.

Our main concern in this paper is the linear stability of the sub-to-super inviscid transonic stationary waves $(\rho(x; 0), u(x; 0))$ constructed in [11]. First, we consider the L^2 - and H^1 - energy for the linearized problem of (1.1) about $(\rho(x; 0), u(x; 0))$. Suppose $u_- > \sqrt{P'(\rho_-)}/2$, then we are able to simultaneously symmetrize the convection matrix (see Proposition 3.1) and establish the non-negative definiteness of the reaction matrix (see Lemmas 3.3 and 3.4) in L^2 - and H^1 energies. However, due to the fact $\rho_x(\pm\infty; 0) = 0$, the energy method

only implies that the L^2 -energy (or some linear combination of L^2 - and H^1 -energies) is decreasing in time variable. This fact does not imply the linear stability. To overcome this difficult, we further provide more delicate estimates and show that the sub-to-super inviscid transonic stationary wave, on any bounded space interval, is linearly L^∞ stable. Note that our result can be generalized to sub-to-super viscous transonic stationary waves if $(\rho(x; \varepsilon), u(x; \varepsilon))$ converges to $(\rho(x; 0), u(x; 0))$ smoothly (see Remark 3.8). Moreover, we emphasize that our result is new and we provide a non-standard technique to study the related stability problems.

The rest of this paper is organized as follows. In Section 2, we briefly recall the construction of the sub-to-super inviscid transonic stationary wave and its viscous profiles by using the geometric singular perturbation approach. In Section 3, our main result, the linearly L^∞ stability of the sub-to-super inviscid transonic stationary wave for the inviscid system (1.1), will be established.

2 Sub-to-super transonic stationary waves

In [11], we showed the existence of the *smooth* sub-to-super inviscid transonic stationary wave for the inviscid system (1.1) and its viscous profiles as stationary waves of (1.4). For the sake of completeness, in this section we present the general geometric singular perturbation setup, sketch the construction and refer to [11] for a detailed proof and other related results. Throughout this paper, we always assume the conditions (A-1) and (A-2) hold.

The steady-state problem of system (1.4) is

$$\begin{cases} (a\rho u)_x = 0, \\ (a\rho u^2)_x + a(P(\rho))_x = \varepsilon(au_x)_x, \end{cases} \quad (2.7)$$

Definition 2.1. Let $(\rho(x), u(x))$ be a bounded smooth solution of (2.7) with $\varepsilon = 0$. We say $(\rho(x), u(x))$ has a viscous profile if, for $\varepsilon > 0$ small, system (2.7) has a solution $(\rho(x; \varepsilon), u(x; \varepsilon))$ such that $(\rho(x; \varepsilon), u(x; \varepsilon)) \rightarrow (\rho(x), u(x))$ as $\varepsilon \rightarrow 0$ uniformly for $x \in \mathbb{R}$.

Remark 2.2. In [11], we showed that, for a bounded smooth solution $(\rho(x), u(x))$, necessarily, $(\rho(x; \varepsilon), u(x; \varepsilon)) \rightarrow (\rho_\pm(\varepsilon), u_\pm(\varepsilon))$ as $x \rightarrow \pm\infty$ for some $(\rho_\pm(\varepsilon), u_\pm(\varepsilon))$ and $\varepsilon > 0$. In Definition 2.1, we allow $(\rho_\pm(\varepsilon), u_\pm(\varepsilon)) \neq (\rho_\pm, u_\pm)$ but require $(\rho_\pm(\varepsilon), u_\pm(\varepsilon)) \rightarrow (\rho_\pm, u_\pm)$ as $\varepsilon \rightarrow 0$.

Let $(\rho(x; \varepsilon), u(x; \varepsilon))$ be a solution of (2.7) and assume $(\rho(x; \varepsilon), u(x; \varepsilon)) \rightarrow (\rho_\pm(\varepsilon), u_\pm(\varepsilon))$ as $x \rightarrow \pm\infty$ and $(\rho_\pm(\varepsilon), u_\pm(\varepsilon)) \rightarrow (\rho_\pm, u_\pm)$ as $\varepsilon \rightarrow 0$. The first equation in (2.7) gives

$$a(x)\rho(x; \varepsilon)u(x; \varepsilon) = a_- \rho_-(\varepsilon)u_-(\varepsilon) = a_+ \rho_+(\varepsilon)u_+(\varepsilon).$$

Let $m_\varepsilon = a_\pm \rho_\pm(\varepsilon) u_\pm(\varepsilon)$ and $m_0 = a_\pm \rho_\pm u_\pm$. For definiteness, we will consider the case $m_\varepsilon \geq 0$. System (2.7) reduces to

$$(m_\varepsilon^2 a^{-1} \rho^{-1})_x + a(P(\rho))_x = \varepsilon(a(m_\varepsilon a^{-1} \rho^{-1})_x)_x. \quad (2.8)$$

If $m_\varepsilon = 0$, system (2.8) has a trivial solution with $\rho = \text{const}$ and $u = 0$. Therefore, we will only consider the case $m_0 > 0$ and $m_\varepsilon > 0$ in the sequel.

2.1 A dynamical system formulation

The problem (2.8) can be examined using the geometric singular perturbation approach of dynamical systems. In order to do so, we will rewrite the equation (2.8) as an *autonomous* system of first order equations. Thus, we introduce the variables:

$$w := \varepsilon a(m_\varepsilon a^{-1} \rho^{-1})_x - m_\varepsilon^2 a^{-1} \rho^{-1} - aP(\rho), \quad \eta_x = 1 - \eta^2 \text{ with } \eta(0) = 0. \quad (2.9)$$

It is obvious that $\eta(x)$ is an increasing function for $-\infty < x < \infty$ and $\eta(\pm\infty) = \pm 1$. We denote the inverse by $x = x(\eta)$ with $-1 \leq \eta \leq 1$ and treat $a(x)$ as a function of η via $x = x(\eta)$. System (2.8) is recast as

$$\begin{cases} \varepsilon \dot{\rho} = -\varepsilon a^{-1} a_x \rho - m_\varepsilon^{-1} \rho^2 (w + m_\varepsilon^2 a^{-1} \rho^{-1} + aP(\rho)), \\ \dot{w} = -a_x P(\rho), \\ \dot{\eta} = 1 - \eta^2, \end{cases} \quad (2.10)$$

where “ $\cdot$ ” means $\frac{d}{dx}$. A naive way to make the system autonomous is to augment the equation $\dot{x} = 1$ instead of $\dot{\eta} = 1 - \eta^2$. In doing so, one has to consider the whole phase space $\mathbb{R}^3$. With the introduction of the variable η , since $\eta \rightarrow \pm 1$ as $x \rightarrow \pm\infty$, we can restrict system (2.10) on the compact portion $\{-1 \leq \eta \leq 1\}$ of η -variable so that the invariant manifold theory ([6, 8]) can be applied directly in our construction of stationary wave solutions.

In terms of the fast scale $\xi = x/\varepsilon$, system (2.10) becomes

$$\begin{cases} \rho' = -\varepsilon a^{-1} a_x \rho - m_\varepsilon^{-1} \rho^2 (w + m_\varepsilon^2 a^{-1} \rho^{-1} + aP(\rho)), \\ w' = -\varepsilon a_x P(\rho), \\ \eta' = \varepsilon(1 - \eta^2), \end{cases} \quad (2.11)$$

where “ $\prime$ ” means $\frac{d}{d\xi}$.

The equilibria of system (2.11) are $B_\pm(\varepsilon) = (\rho_\pm(\varepsilon), w_\pm(\varepsilon), \pm 1)$ where

$$w_\pm(\varepsilon) = -m_\varepsilon^2 a_\pm^{-1} \rho_\pm^{-1}(\varepsilon) - a_\pm P(\rho_\pm(\varepsilon)).$$

Thus the existence of solutions of (2.8) is equivalent to finding an orbit (ρ, w, η) of (2.10) or equivalently (2.11) in $\mathbb{R}^3$ that connects $B_-(\varepsilon)$ to $B_+(\varepsilon)$.

The idea of the geometric singular perturbation theory ([7, 12]) is to first understand the limiting systems of (2.10) and (2.11) at $\varepsilon = 0$ and then piece the information together to make conclusions for $\varepsilon > 0$ small.

2.2 Slow manifold, limiting slow and fast dynamics.

Now, let's consider the limiting slow and fast systems of (2.10) and (2.11). Setting $\varepsilon = 0$ in (2.10) and (2.11), we have the limiting slow and fast systems

$$\begin{cases} 0 = m_0^{-1}\rho^2(w + m_0^2a^{-1}\rho^{-1} + aP(\rho)), \\ \dot{w} = -a_xP(\rho), \\ \dot{\eta} = 1 - \eta^2, \end{cases} \quad (2.12)$$

and

$$\begin{cases} \rho' = -m_0^{-1}\rho^2(w + m_0^2a^{-1}\rho^{-1} + aP(\rho)), \\ w' = 0, \\ \eta' = 0. \end{cases} \quad (2.13)$$

2.2.1 Slow manifold, normal hyperbolicity and turning points

The slow manifold $\mathcal{Z}_0$ is given by the algebraic equation in (2.12)

$$\mathcal{Z}_0 = \{(\rho, w, \eta) : w = -m_0^2a^{-1}\rho^{-1} - aP(\rho)\}, \quad (2.14)$$

and it is the set of equilibria of system (2.13). The linearization of system (2.13) along the slow manifold $\mathcal{Z}_0$ has three eigenvalues

$$0, 0 \text{ and } \lambda = m_0^{-1}a\rho^2(m_0^2a^{-2}\rho^{-2} - P'(\rho)).$$

The two zero eigenvalues correspond to the tangent space of $\mathcal{Z}_0$ and the eigenvalue λ corresponds to the transversal direction to $\mathcal{Z}_0$.

According to the sign of λ , $\mathcal{Z}_0$ is split into $\mathcal{Z}_0 = \mathcal{Z}_0^s \cup T \cup \mathcal{Z}_0^u$ where

$$\begin{aligned} \mathcal{Z}_0^u &= \{(\rho, w, \eta) \in \mathcal{Z}_0 : m_0a^{-1}\rho^{-1} > \sqrt{P'(\rho)}, \rho > 0, \eta \in [-1, 1]\}, \\ \mathcal{Z}_0^s &= \{(\rho, w, \eta) \in \mathcal{Z}_0 : m_0a^{-1}\rho^{-1} < \sqrt{P'(\rho)}, \rho > 0, \eta \in [-1, 1]\}, \\ T &= \{(\rho, w, \eta) \in \mathcal{Z}_0 : m_0a^{-1}\rho^{-1} = \sqrt{P'(\rho)}, \rho > 0, \eta \in [-1, 1]\}. \end{aligned} \quad (2.15)$$

The portion $\mathcal{Z}_0^s$ is (normally) stable, $\mathcal{Z}_0^u$ is (normally) repelling and T is the set of turning points.

Remark 2.3. Recall that $u = m_0a^{-1}\rho^{-1}$. Thus, the set $\mathcal{Z}_0^u$ consists of supersonic states ($u > \sqrt{P'}$), T sonic states ($u = \sqrt{P'}$), and $\mathcal{Z}_0^s$ subsonic states ($u < \sqrt{P'}$). $\square$

It is shown ([11]) that $\mathcal{Z}_0^s$, $\mathcal{Z}_0^u$ and T can be represented as the graphs of functions $\rho = \rho_1(w, \eta)$, $\rho = \rho_2(w, \eta)$ and $(\rho, w) = (\rho_0(w_0(\eta)), w_0(\eta))$, respectively (see Figure 1).

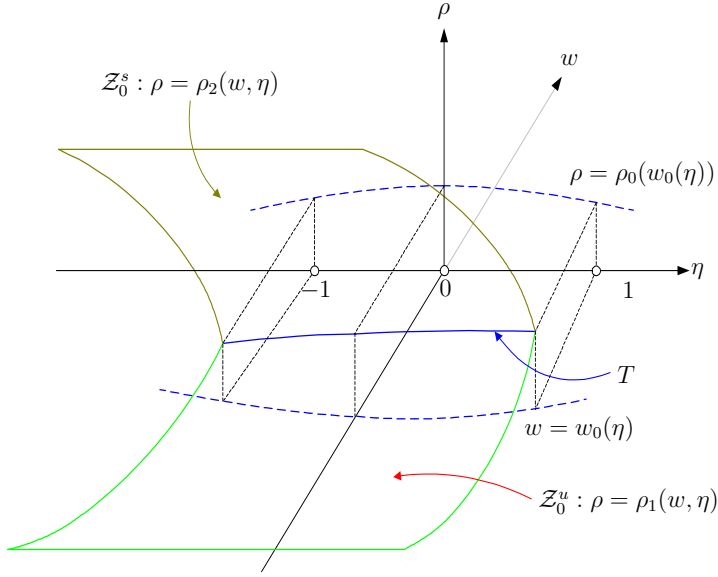


Figure 1: The slow manifold $\mathcal{Z}_0$ and its partition $\mathcal{Z}_0 = \mathcal{Z}_0^s \cup T \cup \mathcal{Z}_0^u$.

2.2.2 Limiting slow dynamics on $\mathcal{Z}_0$

To study the dynamics on the slow manifold $\mathcal{Z}_0$, we use the representation (2.14) to identify $\mathcal{Z}_0$ with the $\rho\eta$ -plane. Then, on the $\rho\eta$ -plane with ρ -axis as the vertical one, T separates the band $\{(\rho, \eta) : \rho > 0, \eta \in [-1, 1]\}$ into two parts: $\mathcal{Z}_0^s$ lies above T and $\mathcal{Z}_0^u$ below T (see Figure 2).

To represent the limiting slow flow (2.12) on $\mathcal{Z}_0$ in terms of the variable (ρ, η) , we differentiate

$$w = -m_0^2 a^{-1} \rho^{-1} - aP(\rho)$$

with respect to x and use system (2.12) to get

$$(P'(\rho) - m_0^2 a^{-2} \rho^{-2}) \dot{\rho} = m_0^2 a_x a^{-3} \rho^{-1}.$$

Note that $P'(\rho) - m_0^2 a^{-2} \rho^{-2} = 0$ on T . Therefore, in the variable (ρ, η) , the limiting slow flow on $\mathcal{Z}_0^s \cup \mathcal{Z}_0^u$ is

$$\dot{\rho} = \frac{m_0^2 a_x a^{-3} \rho^{-1}}{P'(\rho) - m_0^2 a^{-2} \rho^{-2}}, \quad \dot{\eta} = 1 - \eta^2. \quad (2.16)$$

We remark that both $\mathcal{Z}_0^s$ and $\mathcal{Z}_0^u$ are not invariant under system (2.16) and system (2.16) is not defined along T . But, at the turning point $(\rho_0, 0)$ where $m_0^2 a^{-2}(0) \rho_0^{-2} = P'(\rho_0)$, the numerator $m_0^2 a_x a^{-3} \rho^{-1}$ vanishes too since $a_x(0) = 0$. In fact, system (2.16) can be continued to $(\rho_0, 0)$. The turning point $(\rho_0, 0)$ is called a *canard* point and all other turning points are *fold* points ([4, 13]). A

crucial effect of the turning point $(\rho_0, 0)$ on the dynamics for $\varepsilon > 0$ will be given in Theorem 2.7.

The dynamics of (2.16) is completely determined by

Lemma 2.4. *System (2.16) has an integral*

$$I(\rho, \eta) = \int_{\rho_0}^{\rho} \frac{P'(s)}{s} ds + \frac{m_0^2}{2a^2(x(\eta))\rho^2}. \quad (2.17)$$

The following statements can also be easily checked for system (2.16).

Lemma 2.5. *On $\mathcal{Z}_0^s \cap \{-1 \leq \eta < 0\}$ and $\mathcal{Z}_0^u \cap \{0 < \eta \leq 1\}$, the ρ -component of a solution is decreasing; On $\mathcal{Z}_0^s \cap \{0 < \eta \leq 1\}$ and $\mathcal{Z}_0^u \cap \{-1 \leq \eta < 0\}$, the ρ -component is increasing (see Figure 2).*

Proof. On $\mathcal{Z}_0^s \cap \{-1 \leq \eta < 0\}$, due to the assumption (A-2) in (1.6), $P'(\rho) - m^2 a^{-2} \rho^{-2} > 0$ and $a_x < 0$. In view of the ρ -component of system (2.16), we have that the ρ -component of a solution is decreasing. All other statements can be verified in the same way. $\square$

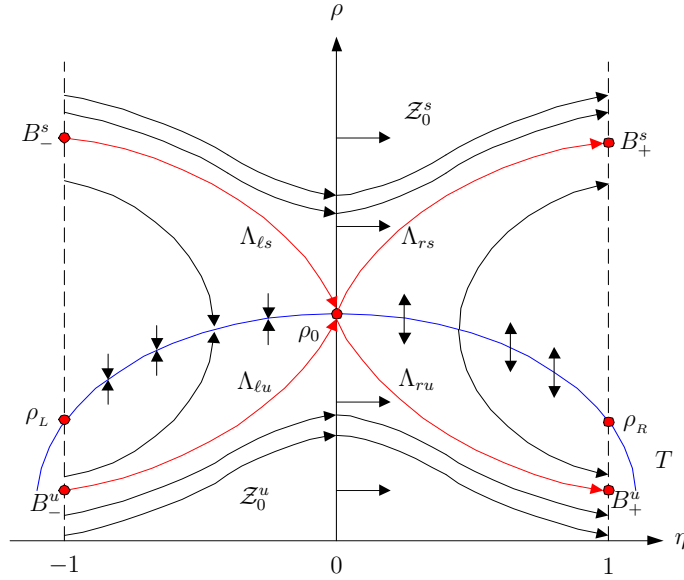


Figure 2: *Standing waves, the smooth sub-to-super transonic wave $\Lambda_{\ell s} \cup \Lambda_{ru}$ and the smooth super-to-sub transonic wave $\Lambda_{\ell u} \cup \Lambda_{rs}$.*

2.2.3 Limiting fast dynamics

The limiting fast dynamics (2.13) is trivial. The variable (w, η) is fixed, $\mathcal{Z}_0$ is the set of equilibria, and ρ will approach $\mathcal{Z}_0^s$ in forward direction and approach $\mathcal{Z}_0^u$ in backward direction. The orbits correspond to shock waves.

2.3 A sub-to-super inviscid transonic orbit and its viscous profile

We now ready to construct sub-to-super inviscid transonic orbits and their viscous profiles. Define ρ_L and η_R by $m_0 = a_- \rho_L \sqrt{P'(\rho_L)}$ and $m_0 = a_+ \rho_R \sqrt{P'(\rho_R)}$; that is, $(\rho_L, -1) = T \cap \{\eta = -1\}$ and $(\rho_R, 1) = T \cap \{\eta = 1\}$ (see Figure 2).

Lemma 2.6. *There are exactly four points $B_\pm^s = (\rho_\pm^s, \pm 1) \in \mathcal{Z}_0^s$ and $B_\pm^u = (\rho_\pm^u, \pm 1) \in \mathcal{Z}_0^u$ with $\rho_-^u < \rho_L < \rho_-^s$ and $\rho_+^u < \rho_R < \rho_+^s$ such that*

$$I(B_\pm^s) = I(B_\pm^u) = I(\rho_0, 0), \quad (2.18)$$

where I is defined in (2.17), and each point is connected with $(\rho_0, 0)$ by the corresponding level curve (see Figure 2).

We denote the singular orbits in $\mathcal{Z}_0^u$ from B_-^u to $(\rho_0, 0)$ by $\Lambda_{\ell u}$, from $(\rho_0, 0)$ to B_+^u by Λ_{ru} , and the singular orbits in $\mathcal{Z}_0^s$ from B_-^s to $(\rho_0, 0)$ by $\Lambda_{\ell s}$, from $(\rho_0, 0)$ to B_+^s by Λ_{rs} . The singular orbit $\mathcal{O}_0 = \Lambda_{\ell s} \cup \Lambda_{ru}$ is the *sub-to-super inviscid transonic stationary wave* of the inviscid gas flow for the given m_0 . The following result in [11] provides the existence of the viscous profile for the sub-to-super transonic stationary wave $\mathcal{O}_0$.

For $\varepsilon > 0$, we now go back to the phase space $\mathbb{R}^3$. Due to the normal hyperbolicity of $\mathcal{Z}_0^s$ and $\mathcal{Z}_0^u$ from (2.15), the invariant manifold theory ([6, 8]) implies the existence of $\mathcal{Z}_\varepsilon^s$ and $\mathcal{Z}_\varepsilon^u$ for $\varepsilon > 0$ small. But $\mathcal{Z}_\varepsilon^s$ and $\mathcal{Z}_\varepsilon^u$ will break in the vicinity of the turning point set T ([19, 20]). If we choose a cross-section $\Sigma = \{(\rho, w, \eta) : \rho = \sigma(w, \eta)\}$ near T , then, for $\varepsilon > 0$, $T_\varepsilon^{s,u} = \mathcal{Z}_\varepsilon^{s,u} \cap \Sigma$ are curves on Σ . Let

$$T_\varepsilon^{s,u} = \{(\rho, w, \eta) : \rho = \sigma(w, \eta), w = w_\varepsilon^{s,u}(\eta)\}.$$

Theorem 2.7. *There exists η_ε , $O(\varepsilon)$ -close to 0, so that $w_\varepsilon^s(\eta_\varepsilon) = w_\varepsilon^u(\eta_\varepsilon) = w_\varepsilon$,*

$$w_\varepsilon^s(\eta) > w_\varepsilon^u(\eta) \text{ for } \eta \in (-1, \eta_\varepsilon) \text{ and } w_\varepsilon^s(\eta) < w_\varepsilon^u(\eta) \text{ for } \eta \in (\eta_\varepsilon, 1).$$

In particular, for $\varepsilon > 0$ small, there is a unique orbit $\mathcal{O}_\varepsilon$ of system (2.10) or equivalently (2.11) such that $\mathcal{O}_\varepsilon \rightarrow \mathcal{O}_0$ as $\varepsilon \rightarrow 0$. See Figure 3.

Proof. See [11, 19]. □

In terms of the original variables, we have, for any fixed $m_0 > 0$, there is a unique smooth sub-to-super transonic stationary wave $(\rho(x; 0), u(x; 0))$ that corresponds to the orbit $\mathcal{O}_0$ above so that

- (i) $\rho_x(x; 0) < 0$, $(\rho(x; 0), u(x; 0)) \rightarrow (\rho_\pm, u_\pm)$ as $x \rightarrow \pm\infty$;
- (ii) $m_0 = a_\pm \rho_\pm u_\pm$, $u(x; 0) < \sqrt{P'(\rho(x; 0))}$ for $x < 0$, $u(x; 0) = \sqrt{P'(\rho(x; 0))}$ at $x = 0$, and $u(x; 0) > \sqrt{P'(\rho(x; 0))}$ for $x > 0$.

Take the derivative with respect to a_- to get

$$\frac{d\rho_-}{da_-} = \frac{m_0 a_-^{-3} \rho_-^{-1}}{P'(\rho_-) - m_0^2 a_-^{-2} \rho_-^{-2}} > 0.$$

Note that $\rho_- \rightarrow \rho_0$ as $a_- \rightarrow a_0$ and hence $\frac{d\rho_-}{da_-} \rightarrow +\infty$ as $a_- \rightarrow a_0$.

Now set $r(a_-) = 2m_0^2 - a_-^2 \rho_-^2 P'(\rho_-)$. Then $r(a_-) \rightarrow m_0^2 > 0$ as $a_- \rightarrow a_0$ and $r(a_-) \rightarrow -\infty$ as $a_- \rightarrow +\infty$. Since

$$\frac{\partial r}{\partial a_-} = -2a_- \rho_-^2 P'(\rho_-) - (2a_-^2 \rho_- P'(\rho_-) + a_-^2 \rho_-^2 P''(\rho_-)) \frac{d\rho_-}{da_-} < 0,$$

we conclude that there exists a unique $a^* = a^*(m_0) > a_0$ such that $r(a^*) = 0$ and $r(a_-) > 0$ provided $a_0 < a_- < a^*$. Note that $r(a_-) > 0$ is equivalent to $u_- > \sqrt{P'(\rho_-)}/2$. The proof is complete. $\square$

3 Linear stability of the sub-to-super inviscid transonic stationary wave

In this section, we first setup the linearized problem of (1.4) with respect to the smooth stationary waves. Using energy method, if $u_- > \sqrt{P'(\rho_-)}/2$, we provide some basic properties and show that the L^2 -energy for the inviscid problem with respect to the sub-to-super inviscid transonic stationary wave is decreasing in the time variable. Then we establish our main result that, the sub-to-super inviscid transonic stationary wave $(\bar{\rho}(x; 0), \bar{u}(x; 0))$ is linearly L^∞ stable on any bounded space interval.

3.1 Setup and basic lemmas

We set up the linearization problem about a general smooth stationary wave of system (1.4).

Set $A(x) := a_x(x) a^{-1}(x)$, then system (1.4) becomes

$$\begin{cases} \rho_t + (\rho u)_x = -A(x) \rho u, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = -A(x) \rho u^2 + \varepsilon A(x) u_x + \varepsilon u_{xx}. \end{cases} \quad (3.19)$$

Let $(\bar{\rho}(x; \varepsilon), \bar{u}(x; \varepsilon))$ be a smooth stationary wave of system (1.4) and set $\bar{U} = (\bar{\rho}, \bar{\rho} \bar{u})^T$,

$$f(\bar{U}) = (\bar{\rho} \bar{u}, \bar{\rho} \bar{u}^2 + P(\bar{\rho}))^T \quad \text{and} \quad g(\bar{U}) = (-\bar{\rho} \bar{u}, -\bar{\rho} \bar{u}^2)^T.$$

Then the linearized equation of system (3.19) with respect to $\bar{U}$ is

$$U_t + (Df(\bar{U}) \cdot U)_x = A(x) Dg(\bar{U}) \cdot U + \varepsilon M_0 U + \varepsilon M_1 U_x + \varepsilon M_2 U_{xx},$$

or

$$U_t = (A(x)Dg(\bar{U}) - (Df(\bar{U}))_x + \varepsilon M_0)U + (\varepsilon M_1 - Df(\bar{U}))U_x + \varepsilon M_2 U_{xx}, \quad (3.20)$$

where $U = (\rho, \rho u)^T$,

$$\begin{aligned} Df(\bar{U}) &= \begin{pmatrix} 0 & 1 \\ -\bar{u}^2 + P'(\bar{\rho}) & 2\bar{u} \end{pmatrix}, \quad Dg(\bar{U}) = \begin{pmatrix} 0 & -1 \\ \bar{u}^2 & -2\bar{u} \end{pmatrix}, \\ M_0 &= \begin{pmatrix} 0 & 0 \\ -A(x)(\bar{\rho}^{-1}\bar{u})_x - (\bar{\rho}^{-1}\bar{u})_{xx} & -A(x)\bar{\rho}^{-2}\bar{\rho}_x - (\bar{\rho}^{-2}\bar{\rho}_x)_x \end{pmatrix}, \\ M_1 &= \begin{pmatrix} 0 & 0 \\ -A(x)\bar{\rho}^{-1}\bar{u} - 2(\bar{\rho}^{-1}\bar{u})_x & A(x)\bar{\rho}^{-1} - 2\bar{\rho}^{-2}\bar{\rho}_x \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ -\bar{\rho}^{-1}\bar{u} & \bar{\rho}^{-1} \end{pmatrix}. \end{aligned}$$

It's easy to see that the eigenvalues $\lambda_1(\bar{U})$, $\lambda_2(\bar{U})$ of $Df(\bar{U})$ and the corresponding matrix representation $Q(\bar{U})$ of the left eigenvectors of $Df(\bar{U})$ are

$$\lambda_1(\bar{U}) = \bar{u} - \sqrt{P'(\bar{\rho})}, \quad \lambda_2(\bar{U}) = \bar{u} + \sqrt{P'(\bar{\rho})},$$

and

$$Q(\bar{U}) = \begin{pmatrix} -\lambda_2(\bar{U}) & 1 \\ -\lambda_1(\bar{U}) & 1 \end{pmatrix} = \begin{pmatrix} -\bar{u} - \sqrt{P'(\bar{\rho})} & 1 \\ -\bar{u} + \sqrt{P'(\bar{\rho})} & 1 \end{pmatrix}.$$

Furthermore, $Q(\bar{U})$ is invertible as long as $\lambda_1(\bar{U}) \neq \lambda_2(\bar{U})$, and

$$\begin{aligned} Q^{-1}(\bar{U}) &= \frac{1}{\lambda_1(\bar{U}) - \lambda_2(\bar{U})} \begin{pmatrix} 1 & -1 \\ \lambda_1(\bar{U}) & -\lambda_2(\bar{U}) \end{pmatrix} \\ &= \frac{-1}{2\sqrt{P'(\bar{\rho})}} \begin{pmatrix} 1 & -1 \\ \bar{u} - \sqrt{P'(\bar{\rho})} & -\bar{u} - \sqrt{P'(\bar{\rho})} \end{pmatrix}. \end{aligned}$$

Let $V := Q(\bar{U})U$ and multiply equation (3.20) by $Q(\bar{U})$ to get

$$V_t = R_0(x; \varepsilon)V + R_1(x; \varepsilon)V_x + R_2(x; \varepsilon)V_{xx}, \quad (3.21)$$

where

$$\begin{aligned} R_0(x; \varepsilon) &:= Q(A(x)Dg - (Df)_x + DfQ^{-1}Q_x)Q^{-1} + \\ &\quad \varepsilon Q(M_0 - M_1Q^{-1}Q_x + 2M_2(Q^{-1}Q_x)^2 - M_2Q^{-1}Q_{xx})Q^{-1}, \\ R_1(x; \varepsilon) &:= -Q(Df - \varepsilon M_1 + 2\varepsilon M_2Q^{-1}Q_x)Q^{-1}, \\ R_2(x; \varepsilon) &:= \varepsilon QM_2Q^{-1}. \end{aligned}$$

A *crucial* fact is the following special structure

Proposition 3.1. *Both $R_1(x; \varepsilon)$ and $R_2(x; \varepsilon)$ are symmetric. In particular,*

$$R_1(x; \varepsilon) = \begin{pmatrix} -\lambda_1(\bar{U}) & 0 \\ 0 & -\lambda_2(\bar{U}) \end{pmatrix} + O(\varepsilon) \quad \text{and} \quad R_2(x; \varepsilon) = \frac{\varepsilon \bar{\rho}^{-1}}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Proof. In fact, a direct computation gives

$$\begin{aligned} R_2(x; \varepsilon) &= -\frac{\varepsilon}{2\sqrt{P'}} \begin{pmatrix} -\bar{u} - \sqrt{P'} & 1 \\ -\bar{u} + \sqrt{P'} & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\bar{\rho}^{-1}\bar{u} & \bar{\rho}^{-1} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ \bar{u} - \sqrt{P'} & -\bar{u} - \sqrt{P'} \end{pmatrix} \\ &= \frac{\varepsilon\bar{\rho}^{-1}}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

For $R_1(x; \varepsilon) = -QDfQ^{-1} + \varepsilon QM_1Q^{-1} - 2\varepsilon QM_2Q^{-1}Q_xQ^{-1}$, we have

$$-QDfQ^{-1} = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}.$$

It thus suffices to show $QM_1Q^{-1} - 2QM_2Q^{-1}Q_xQ^{-1}$ is symmetric. Note that

$$\begin{aligned} QM_1Q^{-1} &= \begin{pmatrix} -\bar{u} - \sqrt{P'} & 1 \\ -\bar{u} + \sqrt{P'} & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -A\bar{\rho}^{-1}\bar{u} - 2(\bar{\rho}^{-1}\bar{u})_x & A\bar{\rho}^{-1} - 2\bar{\rho}^{-2}\bar{\rho}_x \end{pmatrix} Q^{-1} \\ &= \begin{pmatrix} -A\bar{\rho}^{-1}\bar{u} - 2(\bar{\rho}^{-1}\bar{u})_x & A\bar{\rho}^{-1} - 2\bar{\rho}^{-2}\bar{\rho}_x \\ -A\bar{\rho}^{-1}\bar{u} - 2(\bar{\rho}^{-1}\bar{u})_x & A\bar{\rho}^{-1} - 2\bar{\rho}^{-2}\bar{\rho}_x \end{pmatrix} Q^{-1} = \frac{1}{\sqrt{P'}} \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} n_{11} &= n_{21} = (\bar{\rho}^{-1}\bar{u})_x + 1/2A\bar{\rho}^{-1}\sqrt{P'} + \bar{u}\bar{\rho}^{-2}\bar{\rho}_x - \bar{\rho}^{-2}\bar{\rho}_x\sqrt{P'}, \\ n_{12} &= n_{22} = -(\bar{\rho}^{-1}\bar{u})_x + 1/2A\bar{\rho}^{-1}\sqrt{P'} - \bar{u}\bar{\rho}^{-2}\bar{\rho}_x - \bar{\rho}^{-2}\bar{\rho}_x\sqrt{P'}, \end{aligned}$$

and

$$-2QM_2Q^{-1}Q_xQ^{-1} = \frac{1}{\sqrt{P'}} \begin{pmatrix} -\bar{\rho}^{-1}\bar{u}_x & \bar{\rho}^{-1}\bar{u}_x \\ -\bar{\rho}^{-1}\bar{u}_x & \bar{\rho}^{-1}\bar{u}_x \end{pmatrix}.$$

Thus the symmetry of $QM_1Q^{-1} - 2QM_2Q^{-1}Q_xQ^{-1}$ follows from

$$n_{12} + \bar{\rho}^{-1}\bar{u}_x = n_{21} - \bar{\rho}^{-1}\bar{u}_x$$

that can be checked easily. This verifies the statements. $\square$

Next, we consider the regularity of the matrices $R_i(x; 0)$, $i = 0, 1, 2$ by investigating the smoothness of $\bar{\rho}_x(x)$, here $\bar{\rho}(x) = \bar{\rho}(x; 0)$ is the sub-to-super inviscid transonic stationary wave. Since we assume that $a_x(0) = 0$, the right hand side of the first equation in (2.16) is of the form $0/0$ as $x \rightarrow 0$. By L'Hospital's rule,

$$\bar{\rho}_x(0) := \lim_{x \rightarrow 0} \bar{\rho}_x(x) = -\left(\frac{a_{xx}(0)m_0^2a^{-3}(0)\bar{\rho}^{-1}}{P''(\bar{\rho}) + 2m_0^2a^{-2}(0)\bar{\rho}^{-3}} \right)^{1/2}.$$

In view of the first equality in (2.16), we have

$$\bar{\rho}_{xx}(x) = -\bar{\rho}_x \left(\frac{3a_x}{a} + \frac{3\bar{\rho}_x}{\bar{\rho}} - \frac{m_0^2 a_{xx} a^{-3} \bar{\rho} - P''(\bar{\rho}) \bar{\rho}^2 \bar{\rho}_x^2 - 2m_0^2 a^{-2} \bar{\rho}^{-1} \bar{\rho}_x^2}{m_0^2 a_x a^{-3} \bar{\rho}} \right) \quad (3.22)$$

for $x \neq 0$. Note that, by (2.16), the last term in (3.22) is of the form $0/0$ as $x \rightarrow 0$. Applying L'Hospital's rule again (with tedious computations), we obtain

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{m_0^2 a_{xx} a^{-3} \bar{\rho} - P''(\bar{\rho}) \bar{\rho}^2 \bar{\rho}_x^2 - 2m_0^2 a^{-2} \bar{\rho}^{-1} \bar{\rho}_x^2}{m_0^2 a_x a^{-3} \bar{\rho}} \\ &= \lim_{x \rightarrow 0} (m_0^2 a_{xx} a^{-3} \bar{\rho})^{-1} \cdot (m_0^2 a_{xxx} a^{-3} \bar{\rho} + m_0^2 a_{xx} a^{-3} \bar{\rho}_x - P'''(\bar{\rho}) \bar{\rho}^2 \bar{\rho}_x^3 - 2P''(\bar{\rho}) \bar{\rho} \bar{\rho}_x^3 - \\ & \quad 2P''(\bar{\rho}) \bar{\rho}^2 \bar{\rho}_x \bar{\rho}_{xx} + 2m_0^2 a^{-2} \bar{\rho}^{-2} \bar{\rho}_x^3 - 4m_0^2 a^{-2} \bar{\rho}^{-1} \bar{\rho}_x \bar{\rho}_{xx}). \end{aligned} \quad (3.23)$$

Suppose $a_{xx}(0) > 0$, taking limit $x \rightarrow 0$ on both sides of (3.22), then (3.23) implies

$$\begin{aligned} \bar{\rho}_{xx}(0) &:= \lim_{x \rightarrow 0} \bar{\rho}_{xx}(x) \\ &= -\frac{\bar{\rho}_x(0)}{3} \left(\frac{\bar{\rho}_x(0)}{\bar{\rho}(0)} - \frac{\lim_{x \rightarrow 0} a_{xxx}(x)}{a_{xx}(0)} + \bar{\rho}_x^3(0) \frac{P'''(\bar{\rho}(0)) + 3P''(\bar{\rho}(0)) \bar{\rho}^{-1}(0)}{m_0^2 a_{xx}(0) a^{-3}(0) \bar{\rho}^{-1}(0)} \right). \end{aligned}$$

Therefore, the continuity of $a_{xxx}(x)$ at $x = 0$ implies the continuity of $\bar{\rho}_{xx}(x)$ at $x = 0$.

According to the above arguments, the conditions $a_x(0) = 0$, $a_{xx}(0) > 0$, and the continuity of $a_{xxx}(x)$ ensure the continuity of $\bar{\rho}_x(x)$ and $\bar{\rho}_{xx}(x)$, and hence the continuity of $R_0(x; 0)$, $R_{0x}(x; 0)$ and $R_{1x}(x; 0)$. Moreover, if

$$\lim_{x \rightarrow \pm\infty} |a_{xx}(x)/a_x(x)| \leq A \quad \text{for some constant } A > 0, \quad (3.24)$$

that is $a(x)$ converges with exponential rates to $a_{\pm}$ as $x \rightarrow \pm\infty$, then (3.22) implies that

$$|\bar{\rho}_{xx}(x)| \leq C |\bar{\rho}_x(x)| \quad \text{for all } x \in \mathbb{R}, \quad (3.25)$$

where $C > 0$ is a constant. The inequality (3.25) will help us to obtain the energy estimates in the following subsection.

3.2 L^2 - and H^1 - energy estimates for the inviscid case

In this subsection, we study the L^2 - and H^1 - energy estimates for problem (3.21) in inviscid case, i.e. $\varepsilon = 0$. For convenience, we simplify the notations $R_i(x) := R_i(x; 0)$ for $i = 0, 1$ and consider the following initial value problem:

$$\begin{cases} V_t = R_0(x)V + R_1(x)V_x, \\ V(x, 0) = V_0(x), \quad V(\pm\infty, t) = V_x(\pm\infty, t) = 0. \end{cases} \quad (3.26)$$

Let $V = V(x, t) \in H^1(\mathbb{R})$ be the solution of (3.26), we define the L^2 - and H^1 -energy functions by

$$E_0(t) := \|V(\cdot, t)\|_{L^2(\mathbb{R})} \quad \text{and} \quad E_1(t) := \|V_x(\cdot, t)\|_{L^2(\mathbb{R})},$$

respectively. Multiply equation (3.26) by V^T and integrate it with respect to x over $(-\infty, \infty)$ (the limits of the integral will be omitted in the sequel), we get

$$\frac{1}{2} \frac{dE_0^2}{dt} = \int V^T V_t dx = \int V^T R_0 V dx + \int V^T R_1 V_x dx.$$

Since both $R_1(x)$ is symmetric and continuously differentiable, we have

$$\int V^T R_1 V_x dx = -\frac{1}{2} \int V^T R_{1x} V dx.$$

Hence,

$$\frac{1}{2} \frac{dE_0^2}{dt} = \int V^T \left(R_0 - \frac{1}{2} R_{1x} \right) V dx = \int V^T L(x) V dx, \quad (3.27)$$

where

$$L(x) := R_0(x) - \frac{1}{2} R_{1x}(x) = R_0(x) + \frac{1}{2} \text{diag}[(\lambda_1(\bar{U}))_x, (\lambda_2(\bar{U}))_x]_{2 \times 2}.$$

Similarly, multiplying V_{xx}^T and integrating by parts, we get

$$\begin{aligned} \frac{1}{2} \frac{dE_1^2}{dt} &= - \int V_{xx}^T R_0 V dx - \int V_{xx}^T R_1 V_x dx \\ &= \int V_x^T \left(R_0 + \frac{1}{2} R_{1x} \right) V_x dx + \int V_x^T R_{0x} V dx \\ &= \int V_x^T H(x) V_x dx + \int V_x^T R_{0x} V dx, \end{aligned} \quad (3.28)$$

where

$$H(x) := R_0(x) + \frac{1}{2} R_{1x}(x) = R_0(x) - \frac{1}{2} \text{diag}[(\lambda_1(\bar{U}))_x, (\lambda_2(\bar{U}))_x]_{2 \times 2}.$$

Recall from Lemma 2.8 that, for fixed geometry of nozzle, there exists $m^* > 0$ depending on $a(0)$ and $a_- = a(-\infty)$ only (not the whole geometry of the nozzle) so that if $(\bar{\rho}(x), \bar{u}(x))$ is a sub-to-super transonic stationary wave with $a_- \bar{\rho}_- \bar{u}_- = m_0 < m^*$ then $2\bar{u}_-^2 > P'(\bar{\rho}_-)$. In the following, we introduce several important lemmas related to sub-to-super inviscid transonic stationary waves with this condition.

Lemma 3.2. *Let $(\bar{\rho}(x), \bar{u}(x))$ be the sub-to-super inviscid transonic stationary wave. If $2\bar{u}_-^2 > P'(\bar{\rho}_-)$, then, for all x ,*

$$\begin{aligned} \ell_1(x) &:= 4\bar{u}^6 - 4\bar{u}^4 P'(\bar{\rho}) + 3\bar{u}^2 (P'(\bar{\rho}))^2 - (P'(\bar{\rho}))^3 > 0, \\ \ell_2(x) &:= 2\bar{u}^3 - 2\bar{u}^2 (P'(\bar{\rho}))^{1/2} + (P'(\bar{\rho}))^{3/2} > 0, \\ \ell_3(x) &:= 2\bar{u}^3 + 2\bar{u}^2 (P'(\bar{\rho}))^{1/2} - (P'(\bar{\rho}))^{3/2} > 0. \end{aligned}$$

Proof. First of all, by Young's inequality, we have

$$2\bar{u}^3 + (P'(\bar{\rho}))^{3/2} \geq 3\bar{u}^2(P'(\bar{\rho}))^{1/2} \quad \text{or} \quad \frac{2}{3} (2\bar{u}^3 + (P'(\bar{\rho}))^{3/2}) \geq 2\bar{u}^2(P'(\bar{\rho}))^{1/2}.$$

Thus, $\ell_3(x) > 0$ without the assumption that $2\bar{u}_-^2 > P'(\bar{\rho})$. Note that $\ell_1(x) = \ell_2(x)\ell_3(x) - \bar{u}^2(P'(\bar{\rho}))^2$. It follows that $\ell_1(x) > 0$ implies $\ell_2(x) > 0$. It thus suffices to show that $\ell_1(x) > 0$. Now

$$\ell_1(x) = (2\bar{u}^2 - P'(\bar{\rho})) (2\bar{u}^4 - \bar{u}^2 P'(\bar{\rho}) + (P'(\bar{\rho}))^2).$$

Therefore, $\ell_1(x) > 0$ if $2\bar{u}^2 - P'(\bar{\rho}) > 0$ for all x . It follows from (2.7) with $\varepsilon = 0$ that $\bar{u}_x = -\bar{u}^{-1}\bar{\rho}^{-1}P'(\bar{\rho})\bar{\rho}_x$. Hence,

$$\frac{d}{dx}(2\bar{u}^2 - P'(\bar{\rho})) = -\bar{\rho}_x \left(\frac{4P'(\bar{\rho})}{\bar{\rho}} + P''(\bar{\rho}) \right) > 0.$$

Therefore, $\ell_1(x) > 0$ if $2\bar{u}_-^2 - P'(\bar{\rho}_-) > 0$. The proof is complete. $\square$

Lemma 3.3. *Let $(\bar{\rho}(x), \bar{u}(x))$ be the sub-to-super inviscid transonic stationary wave with $2\bar{u}_-^2 > P'(\bar{\rho}_-)$. Then, there exists a constant $\alpha > 0$ such that*

$$V^T L(x) V < \alpha \bar{\rho}_x |V|^2 \quad \text{for any } V \in \mathbb{R}^2.$$

Proof. First, it follows from (2.7) with $\varepsilon = 0$ that

$$a\bar{\rho}_x\bar{u} + a\bar{\rho}\bar{u}_x + a_x\bar{\rho}\bar{u} = 0 \quad \text{and} \quad \bar{\rho}\bar{u}\bar{u}_x + P'(\bar{\rho})\bar{\rho}_x = 0.$$

Thus,

$$\begin{aligned} A(x) &= -(\bar{\rho}^{-1}\bar{\rho}_x + \bar{u}^{-1}\bar{u}_x) = -\bar{\rho}^{-1}\bar{u}^{-2}\bar{\rho}_x(\bar{u}^2 - P'(\bar{\rho})), \\ \lambda_{1x} &= -\bar{\rho}_x \left(\frac{P'(\bar{\rho})}{\bar{\rho}\bar{u}} + \frac{P''(\bar{\rho})}{2\sqrt{P'(\bar{\rho})}} \right), \quad \lambda_{2x} = -\bar{\rho}_x \left(\frac{P'(\bar{\rho})}{\bar{\rho}\bar{u}} - \frac{P''(\bar{\rho})}{2\sqrt{P'(\bar{\rho})}} \right). \end{aligned} \quad (3.29)$$

Recall that $R_0(x) = A(x)QDgQ^{-1} - Q(Df)_xQ^{-1} + QDfQ^{-1}Q_xQ^{-1}$. We have

$$\begin{aligned} A(x)QDgQ^{-1} &= A(x) \begin{pmatrix} \frac{P'}{\lambda_1 - \lambda_2} - \lambda_1 & -\frac{P'}{\lambda_1 - \lambda_2} \\ \frac{P'}{\lambda_1 - \lambda_2} & -\frac{P'}{\lambda_1 - \lambda_2} - \lambda_2 \end{pmatrix}, \\ Q(Df)_xQ^{-1} &= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1(\lambda_1 + \lambda_2)_x - (\lambda_1\lambda_2)_x & (\lambda_1\lambda_2)_x - \lambda_2(\lambda_1 + \lambda_2)_x \\ \lambda_1(\lambda_1 + \lambda_2)_x - (\lambda_1\lambda_2)_x & (\lambda_1\lambda_2)_x - \lambda_2(\lambda_1 + \lambda_2)_x \end{pmatrix}, \\ QDfQ^{-1}Q_xQ^{-1} &= -\frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1\lambda_{2x} & -\lambda_1\lambda_{1x} \\ \lambda_2\lambda_{1x} & -\lambda_2\lambda_{2x} \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned}
R_0(x) &= \begin{pmatrix} \frac{AP'(\bar{\rho}) - \lambda_1 \lambda_{2x}}{\lambda_1 - \lambda_2} - A\lambda_1 - \lambda_{1x} & \frac{\lambda_2 \lambda_{2x} - AP'(\bar{\rho})}{\lambda_1 - \lambda_2} \\ \frac{AP'(\bar{\rho}) - \lambda_1 \lambda_{1x}}{\lambda_1 - \lambda_2} & \frac{\lambda_2 \lambda_{1x} - AP'(\bar{\rho})}{\lambda_1 - \lambda_2} - A\lambda_2 - \lambda_{2x} \end{pmatrix}, \\
L(x) &= R_0(x) + \frac{1}{2} \text{diag}[\lambda_{1x}, \lambda_{2x}]_{2 \times 2} \\
&= \begin{pmatrix} \frac{AP'(\bar{\rho}) - \lambda_1 \lambda_{2x}}{\lambda_1 - \lambda_2} - A\lambda_1 - \frac{\lambda_{1x}}{2} & \frac{\lambda_2 \lambda_{2x} - AP'(\bar{\rho})}{\lambda_1 - \lambda_2} \\ \frac{AP'(\bar{\rho}) - \lambda_1 \lambda_{1x}}{\lambda_1 - \lambda_2} & \frac{\lambda_2 \lambda_{1x} - AP'(\bar{\rho})}{\lambda_1 - \lambda_2} - A\lambda_2 - \frac{\lambda_{2x}}{2} \end{pmatrix}.
\end{aligned}$$

With a careful computation, we obtain that $L(x) = \bar{\rho}_x K = \bar{\rho}_x [k_{ij}(x)]_{2 \times 2}$ where

$$\begin{aligned}
k_{11}(x) &= \frac{1}{\bar{\rho}} \left(\bar{u} - \sqrt{P'(\bar{\rho})} \right) + \frac{P'(\bar{\rho}) \sqrt{P'(\bar{\rho})}}{2\bar{\rho}\bar{u}^2} + \frac{\bar{u}P''(\bar{\rho})}{4P'(\bar{\rho})} \\
&= \frac{1}{2\bar{\rho}\bar{u}^2} (2\bar{u}^3 - 2\bar{u}^2(P'(\bar{\rho}))^{1/2} + (P'(\bar{\rho}))^{3/2}) + \frac{\bar{u}P''(\bar{\rho})}{4P'(\bar{\rho})}, \\
k_{12}(x) &= \left(\bar{u} + \sqrt{P'(\bar{\rho})} \right) \left(\frac{P'(\bar{\rho})}{2\bar{\rho}\bar{u}^2} - \frac{P''(\bar{\rho})}{4P'(\bar{\rho})} \right), \\
k_{21}(x) &= \left(\bar{u} - \sqrt{P'(\bar{\rho})} \right) \left(\frac{P'(\bar{\rho})}{2\bar{\rho}\bar{u}^2} - \frac{P''(\bar{\rho})}{4P'(\bar{\rho})} \right), \\
k_{22}(x) &= \frac{1}{\bar{\rho}} (\bar{u} + \sqrt{P'(\bar{\rho})}) - \frac{P'(\bar{\rho}) \sqrt{P'(\bar{\rho})}}{2\bar{\rho}\bar{u}^2} + \frac{\bar{u}P''(\bar{\rho})}{4P'(\bar{\rho})} \\
&= \frac{1}{2\bar{\rho}\bar{u}^2} (2\bar{u}^3 + 2\bar{u}^2(P'(\bar{\rho}))^{1/2} - (P'(\bar{\rho}))^{3/2}) + \frac{\bar{u}P''(\bar{\rho})}{4P'(\bar{\rho})}.
\end{aligned}$$

The inequalities $\ell_2(x) > 0$ and $\ell_3(x) > 0$ in Lemma 3.2 imply $k_{11}(x) > 0$ and $k_{22}(x) > 0$. Therefore, the matrix K is positive definite if

$$\Delta(K) := 4k_{11}(x)k_{22}(x) - (k_{12}(x) + k_{21}(x))^2 > 0. \quad (3.30)$$

Note that, with $\ell_1(x)$ in Lemma 3.2,

$$\begin{aligned}
4k_{11}(x)k_{22}(x) &= \frac{1}{\bar{\rho}^2\bar{u}^4} (\ell_1(x) + \bar{u}^2(P'(\bar{\rho}))^2) + \frac{2\bar{u}^2P''(\bar{\rho})}{\bar{\rho}P'(\bar{\rho})} + \frac{\bar{u}^2(P''(\bar{\rho}))^2}{4(P'(\bar{\rho}))^2}, \\
(k_{12}(x) + k_{21}(x))^2 &= \left(\frac{P'(\bar{\rho})}{\bar{\rho}\bar{u}} - \frac{\bar{u}P''(\bar{\rho})}{2P'(\bar{\rho})} \right)^2 = \frac{(P'(\bar{\rho}))^2}{\bar{\rho}^2\bar{u}^2} - \frac{P''(\bar{\rho})}{\bar{\rho}} + \frac{\bar{u}^2(P''(\bar{\rho}))^2}{4(P'(\bar{\rho}))^2}.
\end{aligned}$$

Therefore,

$$\Delta(K) = \frac{\ell_1(x)}{\bar{\rho}^2\bar{u}^4} + \frac{2\bar{u}^2P''(\bar{\rho})}{\bar{\rho}P'(\bar{\rho})} + \frac{P''(\bar{\rho})}{\bar{\rho}} > 0.$$

Thus K is positive definite. This completes the proof. $\square$

Denote $\kappa_0 = 1 + \sqrt{1 + \sqrt{2}}$. Then we have

Lemma 3.4. *Let $(\bar{\rho}(x), \bar{u}(x))$ be the sub-to-super inviscid transonic stationary wave with $2\bar{u}^2_- > P'(\bar{\rho}_-)$. Under the further condition that $\kappa_0^2 P'(\bar{\rho}) \geq \bar{\rho} P''(\bar{\rho})$, there there a constant $d > 0$ such that*

$$V^T H(x) V < -2d\bar{\rho}_x |V|^2 \quad \text{for any } V \in \mathbb{R}^2.$$

Proof. It's clear that

$$H(x) = L(x) - \text{diag}[(\lambda_1(\bar{U}))_x, (\lambda_2(\bar{U}))_x]_{2 \times 2}.$$

Let $H(x) = \bar{\rho}_x [\xi_{ij}(x)]_{2 \times 2}$. Then, $\xi_{12}(x) = k_{12}(x)$, $\xi_{21}(x) = k_{21}(x)$, and

$$\begin{aligned} \xi_{11}(x) &= k_{11}(x) + \left(\frac{P'}{\bar{\rho}\bar{u}} + \frac{P''}{2\sqrt{P'}} \right) > 0, \\ \xi_{22}(x) &= k_{22}(x) + \left(\frac{P'}{\bar{\rho}\bar{u}} - \frac{P''}{2\sqrt{P'}} \right) \\ &= \frac{1}{2\bar{\rho}\bar{u}^2} (2\bar{u}^3 + 2\bar{u}^2(P')^{1/2} - (P')^{3/2}) + \left(\frac{\bar{u}P''}{4P'} + \frac{P'}{\bar{\rho}\bar{u}} \right) - \frac{P''}{2\sqrt{P'}} \\ &\geq \frac{\bar{u}}{\bar{\rho}} + \sqrt{\frac{P''}{\bar{\rho}}} - \frac{P''}{2\sqrt{P'}} > \frac{\sqrt{P'}}{\sqrt{2}\bar{\rho}} + \sqrt{\frac{P''}{\bar{\rho}}} - \frac{P''}{2\sqrt{P'}} \\ &= \frac{1}{\sqrt{2}\bar{\rho}\sqrt{P'}} \left(P' + \sqrt{2\bar{\rho}P''}\sqrt{P'} - \frac{\bar{\rho}P''}{\sqrt{2}} \right) \\ &= \frac{1}{\sqrt{2}\bar{\rho}\sqrt{P'}} \left(\sqrt{P'} - \frac{\sqrt{\bar{\rho}P''}}{\kappa_0} \right) \left(\sqrt{P'} + \kappa_0 \frac{\sqrt{\bar{\rho}P''}}{\sqrt{2}} \right) \geq 0, \end{aligned}$$

since $\kappa_0^2 P'(\bar{\rho}) \geq \bar{\rho} P''(\bar{\rho})$. Furthermore, from

$$\begin{aligned} k_{22} - k_{11} &= \frac{1}{\bar{\rho}\bar{u}^2} (2\bar{u}^2(P')^{1/2} - (P')^{3/2}) > 0, \\ k_{11} + k_{22} &= \frac{2\bar{u}}{\bar{\rho}} + \frac{\bar{u}P''}{2P'} > \frac{\bar{u}P''}{2P'}, \quad \Delta(K) > \frac{P''}{\bar{\rho}}, \end{aligned}$$

we get

$$\begin{aligned} &4\xi_{11}(x)\xi_{22}(x) - (\xi_{12}(x) + \xi_{21}(x))^2 \\ &= \Delta(K) + 4(k_{11} + k_{22})\frac{P'}{\bar{\rho}\bar{u}} + 2(k_{22} - k_{11})\frac{P''}{P'^{1/2}} + \frac{P'^2}{\bar{\rho}^2\bar{u}^2} - \frac{P''^2}{4P'} \\ &> \frac{P''}{\bar{\rho}} + \frac{2P''}{\bar{\rho}} - \frac{P''^2}{4P'} = \frac{(12P' - \bar{\rho}P'')P''}{4\bar{\rho}P'} > 0, \end{aligned}$$

since $12P' > \kappa_0^2 P' \geq \bar{\rho} P''$. Thus the matrix $H(x)$ is negative definite. $\square$

Remark 3.5. *The condition $\kappa_0^2 P'(\bar{\rho}) \geq \bar{\rho} P''(\bar{\rho})$ in Lemma 3.4 holds true for the usual γ -law of pressure P . $\square$*

Following the results of Lemmas 3.3 and 3.4, we obtain the energy estimates.

Proposition 3.6. *Assume $\varepsilon \geq 0$ and small enough. Let $(\bar{\rho}(x), \bar{u}(x))$ be the sub-to-super inviscid transonic stationary wave with $2\bar{u}_-^2 > P'(\bar{\rho}_-)$.*

- (1) $E_0(t)$ is decreasing in t .
- (2) If $\kappa_0^2 P'(\bar{\rho}) \geq \bar{\rho} P''(\bar{\rho})$, then there exists a constant $C_1 > 0$ such that $C_1 E_0^2(t) + E_1^2(t)$ is decreasing in t .

Proof. (1) According to Lemma 3.3, we have

$$\frac{1}{2} \frac{dE_0}{dt} = \int V^T L(x) V dx < -\alpha \int |\bar{\rho}_x| |V|^2 dx < 0. \quad (3.31)$$

(2) From the proof of Lemma 3.3, $R_0(x)$ has the form $R_0(x) = \bar{\rho}_x [\eta_{ij}]_{2 \times 2}$. Then, for all $x \in \mathbb{R}$, the inequality (3.25) implies that

$$\|R_{0x}(x)\| \leq |\bar{\rho}_{xx}(x)| \cdot \|[\eta_{ij}]_{2 \times 2}\| + |\bar{\rho}_x(x)| \cdot \left\| \frac{d}{dx} [\eta_{ij}]_{2 \times 2} \right\| \leq \bar{C} |\bar{\rho}_x(x)|$$

for some constant $\bar{C} > 0$. Therefore, by Lemma 3.4 and (3.25), we have

$$\begin{aligned} \frac{1}{2} \frac{dE_1^2}{dt} &= \int V_x^T H(x) V_x dx + \int V_x^T R_{0x}(x) V dx \\ &< -2d \int_{-\infty}^{\infty} |V_x|^2 |\bar{\rho}_x| dx + d \int_{-\infty}^{\infty} |V_x|^2 |\bar{\rho}_x| dx + C_2 \int_{-\infty}^{\infty} |V|^2 |\bar{\rho}_x| dx \\ &= -d \int_{-\infty}^{\infty} |V_x|^2 |\bar{\rho}_x| dx + C_2 \int_{-\infty}^{\infty} |V|^2 |\bar{\rho}_x| dx \end{aligned} \quad (3.32)$$

for some constant $C_2 > 0$. Let us choose a $C_1 > 0$ such that $\alpha C_1 > C_2$. Suppose $\varepsilon = 0$, then $C_1 \times (3.31) + (3.32)$ gives

$$\frac{1}{2} \frac{d\{C_1 E_0^2 + E_1^2\}}{dt} \leq -(C_1 \alpha - C_2) \int_{-\infty}^{\infty} |V|^2 |\bar{\rho}_x| dx - d \int_{-\infty}^{\infty} |V_x|^2 |\bar{\rho}_x| dx < 0.$$

The proof is complete. $\square$

From the results of Proposition 3.6, we only know that $E_0(t)$ and $C_1 E_0^2(t) + E_1^2(t)$ are strictly decreasing in t . However, this fact does not imply that $E_0(t)$ and $E_1(t)$ are decreasing to zero. Thus, instead of considering the L^2 and H^1 linear stabilities in the whole space domain, we consider bounded space interval and show that the sub-to-super inviscid stationary wave is linearly L^∞ stable in next subsection.

3.3 L^∞ linear stability for the sub-to-super inviscid stationary wave

Now we investigate the L^∞ linear stabilities on any bounded space interval for the smooth sub-to-super inviscid stationary wave. Let $M > 0$ be a fixed constant. Suppose $V(x, t)$ satisfies (3.26), for any $\delta > 0$ we denote the set A_δ by

$$A_\delta := \left\{ t \geq 0 : \int_{-M}^M |V(x, t)|^2 dx \geq \delta E_0^2(t) \right\}.$$

First, we claim that A_δ is a well-defined measurable set in $[0, \infty)$. Note that $E_0^2(t)$ is a decreasing function on $[0, \infty)$. Integrating equation (3.26) from $-M$ to M , the symmetry property of $R_1(x)$ gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|V(\cdot, t)\|_{L^2(-M, M)}^2 &= \int_{-M}^M V^T R_0(x) V dx + \int_{-M}^M V^T R_1(x) V_x dx \\ &= \frac{1}{2} V^T R_1 V \Big|_{-M}^M + \int_{-M}^M V^T L(x) V dx. \end{aligned}$$

Then Lemma 3.3 implies

$$\frac{d}{dt} \|V(\cdot, t)\|_{L^2(-M, M)}^2 \leq V^T R_1(x) V \Big|_{-M}^M. \quad (3.33)$$

By part (2) of Proposition 3.6, we observe that $V(\cdot, t) \in L^\infty(0, \infty; H^1(\mathbb{R}))$. Thus, $\|V(\cdot, t)\|_{L^2}$ is bounded on $\mathbb{R}^+$. Moreover, Sobolev embedding theorem gives $V(\cdot, t) \in C^{0,1/2}(\mathbb{R})$ and $\|V(\cdot, t)\|_{C^{0,1/2}} \leq C_3$ for $t \in \mathbb{R}^+$. For $t > 0$, we choose $\bar{x}_t \in [-M, M]$ such that

$$|V(\bar{x}_t, t)| = \min_{x \in [-M, M]} |V(x, t)|.$$

The boundedness of $\|V(\cdot, t)\|_{L^2}$ implies that $|V(\bar{x}_t, t)| \leq C_4$ for some constant $C_4 > 0$. Hence, we have

$$|V(x, t)| \leq |V(\bar{x}_t, t)| + \|V(\cdot, t)\|_{C^{0,1/2}} |x - \bar{x}_t|^{1/2} \leq C_4 + (2M)^{1/2} C_3$$

for $(x, t) \in [-M, M] \times \mathbb{R}^+$. Then it follows from (3.33), the boundedness of $V(x, t)$ and $(\bar{\rho}(x), \bar{u}(x))$ that $\|V(\cdot, t)\|_{L^2(-M, M)}^2$ is Lipschitz continuous on $[0, \infty)$. Therefore, A_δ is a measurable set in $[0, \infty)$ and our claim follows.

Next, we claim that, for any constant $0 < M < \infty$,

$$\int_{-M}^M |V(x, t)|^2 dx \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (3.34)$$

It suffices to prove for the case $E_0(t) \geq C_5$ for some constant $C_5 > 0$. Otherwise, the inequality (3.31) gives $E_0^2(t) \rightarrow 0$ as $t \rightarrow \infty$, which trivially implies the

desired result (3.34). For the case $E_0(t) \geq C_5$ for some constant $C_5 > 0$, we first show that $|A_\delta| < \infty$ for any $\delta > 0$, where $|\cdot|$ denotes the usual Lebesgue measure on $[0, \infty)$. If not, then there exists a constant $\delta > 0$ such that $|A_\delta| = \infty$. Since $|\bar{\rho}_x(x)| \geq C_6$ for some constant $C_6 > 0$ and $x \in [-M, M]$, then for $t \in A_\delta$, the inequality (3.31) gives

$$\frac{1}{2} \frac{dE_0^2(t)}{dt} \leq -C_6 \alpha \int_{-M}^M |V(x, t)|^2 dx \leq -C_6 \alpha \delta E_0^2(t). \quad (3.35)$$

Thus, (3.35) tells that there exists a constant $C_7 > 0$ such that

$$\frac{1}{2} \frac{dE_0^2}{dt} \leq -C_7 < 0 \quad \text{for } t \in A_\delta,$$

which implies

$$E_0^2(t) \leq E^2(0) - C_7 |A_\delta \cap [0, t]| < 0$$

for $t > T$ provided T is large enough. This fact contradicts to $E_0^2(t) \geq 0$. Hence, $|A_\delta| < \infty$ for any $\delta > 0$. Let $T_\delta > 0$ be large enough such that $|A_\delta \cap [T_\delta, \infty)| < \delta$. By the definition of A_δ , we know that

$$\int_{-M}^M |V(x, t)|^2 dx < \delta E_0^2(t) \leq \delta E_0^2(0)$$

for $t > T_\delta$ except for $t \in A_\delta \cap [T_\delta, \infty)$. Since $h(t) := \|V(\cdot, t)\|_{L^2(-M, M)}^2$ is Lipschitz continuous on $[0, \infty)$, we have

$$\int_{-M}^M |V(x, t)|^2 dx \leq \delta \{E_0^2(0) + \|h\|_{C^{0,1}}\} \quad \text{for } t \geq T_\delta,$$

which gives the desired result (3.34).

Now, for $t > 0$, we choose $\bar{x}_t \in [-M, M]$ such that

$$|V(\bar{x}_t, t)| = \min_{x \in [-M, M]} |V(x, t)|.$$

The fact that $V(\cdot, t) \in C^{0,1/2}(\mathbb{R})$ and (3.34) indicate that

$$|V(\bar{x}_t, t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (3.36)$$

According to the result of part (2) of Lemma 3.6, we have

$$\int_{-M}^M |V_x(x, t)|^2 dx \leq E_1^2(t; 0) \leq C_1 E_0^2(0) + E_1^2(0) \quad \text{for all } t \geq 0. \quad (3.37)$$

Therefore, for $x \in [-M, M]$, we have

$$\begin{aligned} |V(x, t)|^2 &= |V(\bar{x}_t, t)|^2 + 2 \int_{\bar{x}_t}^x V^T(y, t) V_x(y, t) dy \\ &\leq |V(\bar{x}_t, t)|^2 + 2 \left(\int_{-M}^M |V(y, t)|^2 dy \right)^{1/2} \left(\int_{-M}^M |V_x(y, t)|^2 dy \right)^{1/2}. \end{aligned}$$

It follows from (3.36), (3.34), and (3.37) that

$$\sup_{x \in [-M, M]} |V(x, t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

In conclusion, we obtain the following stability result.

Theorem 3.7. *Let $(\bar{\rho}(x), \bar{u}(x))$ be the smooth sub-to-super inviscid transonic stationary wave of (1.4). Assume $a(x)$ satisfies $a_{xx}(0) > 0$ and condition (3.25),*

$$\bar{u}_- > \sqrt{P'(\bar{\rho}_-)/2} \quad \text{and} \quad \kappa_0^2 P'(\bar{\rho}) \geq \bar{\rho} P''(\bar{\rho}),$$

where $\kappa_0 = 1 + \sqrt{1 + \sqrt{2}}$. Then, on any bounded space interval, $(\bar{\rho}(x), \bar{u}(x))$ is linearly L^∞ -stable.

Remark 3.8. *According to the geometric singular perturbation theory, we know that the sub-to-super viscous stationary waves of (1.4) $\bar{U}(x; \varepsilon) = (\bar{\rho}(x; \varepsilon), \bar{u}(x; \varepsilon))$ converge to the sub-to-super inviscid stationary wave $\bar{U}(x; 0) = (\bar{\rho}(x; 0), \bar{u}(x; 0))$ as ε tends to 0. However, it's unknown that $\bar{U}_x(x; \varepsilon)$ and $\bar{U}_{xx}(x; \varepsilon)$ converge to $\bar{U}_x(x; 0)$ and $\bar{U}_{xx}(x; 0)$ or not. If yes, by the same arguments, we can obtain the similar result of Theorem 3.7 for the case of sub-to-super viscous stationary waves.*

References

- [1] S. R. Chakravarthy and S. Osher, Numerical experiments with the Osher upwind scheme for the Euler equations, *AIAA J.* **21** (1983), pp. 1241-1248.
- [2] R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, Interscience, New York, 1948.
- [3] C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, 2nd edition, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **325**, Springer-Verlag, Berlin, 2005.
- [4] F. Dumortier and R. Roussarie, *Canard cycles and center manifolds*, with an appendix by C. Z. Li, *Mem. Amer. Math. Soc.* **121** (1996), no. 577.
- [5] P. Embid, J. Goodman and A. Majda, Multiple steady states for 1-D transonic flow, *SIAM J. Sci. Stat. Comput.* **5** (1984), pp. 21-41.
- [6] N. Fenichel, Persistence and smoothness of invariant manifolds and flows, *Indiana Univ. Math. J.* **21** (1971), pp. 193-226.
- [7] N. Fenichel, Geometric singular perturbation theory for ordinary differential equations, *J. Differential Equations* **31** (1979), pp. 53-98.

- [8] M. Hirsch, C. Pugh, and M. Shub, *Invariant Manifolds*, Lecture Notes in Math. **583**, Springer-Verlag, New York, 1976.
- [9] J. M. Hong, C.-H. Hsu and B.-C. Huang, Existence and uniqueness of generalized stationary waves for viscous gas flow through a nozzle with discontinuous cross section, *J. Differential Equations* **253** (2012), pp. 1088-1110.
- [10] J. M. Hong, C.-H. Hsu and W. Liu, Viscous standing asymptotic states of isentropic compressible flows through a nozzle, *Arch. Ration. Mech. Anal.* **196** (2010), pp. 575-597.
- [11] J. M. Hong, C.-H. Hsu and W. Liu, Viscous standing asymptotic states of transonic flow through a nozzle of varying area, *J. Differential Equations* **248** (2010), pp. 50-76.
- [12] C.K.R.T. Jones, *Geometric singular perturbation theory* in Dynamical Systems (Montecatini Terme, 1994), Lecture Notes in Math. **1609**, Springer-Verlag, Berlin, 1995, pp. 44-118.
- [13] M. Krupa and P. Szmolyan, Extending geometric singular perturbation theory to nonhyperbolic points-fold and canard points in two dimensions, *SIAM J. Math. Anal.* **33** (2001), pp. 286-314.
- [14] P. D. Lax, Hyperbolic system of conservation laws, II, *Comm. Pure Appl. Math.* **10** (1957), pp. 537-566.
- [15] H. W. Liepmann and A. Roshko, *Elementary of Gas Dynamics*, GALCIT Aeronautical Series, New York: Wiley, 1957.
- [16] T.-P. Liu, Quasilinear hyperbolic system, *Comm. Math. Phys.* **68** (1979), pp. 141-172.
- [17] T.-P. Liu, Transonic gas flow in a nozzle of varying area, *Arch. Ration. Mech. Anal.* **80** (1982), pp. 1-18.
- [18] T.-P. Liu, Nonlinear stability and instability of transonic flows through a nozzle. *Comm. Math. Phys.* **83** (1982), pp. 243-260.
- [19] P. Szmolyan and M. Wechselberger, Canards in $\mathbb{R}^3$, *J. Differential Equations* **177** (2001), pp. 419-453.
- [20] P. Szmolyan and M. Wechselberger, Relaxation oscillations in $\mathbb{R}^3$, *J. Differential Equations* **200** (2004), pp. 69-104.
- [21] G. R. Shubin, A. B. Stephens and H. Glaz, Steady shock tracking and Newton's method applied to one-dimensional nozzle flow, *J. Comput. Phys.* **39** (1980), pp. 364-374.

- [22] D. Serre, *Systems of conservation laws. 1. Hyperbolicity, entropies, shock waves*, Translated from the 1996 French original by I. N. Sneddon, Cambridge Univ. Press, Cambridge, 1999.
- [23] D. Serre, *Systems of conservation laws. 2. Geometric structures, oscillations, and initial-boundary value problems*, Translated from the 1996 French original by I. N. Sneddon. Cambridge Univ. Press, Cambridge, 2000.
- [24] D. H. Smith, Non-uniqueness and multi-shock solutions for transonic flows, *IMA J. Appl. Math.* **71** (2007), pp. 120-132.